Yo-Yo-Math

A book for playing mathematics

Diego Sandoval

Yo-Yo-Math

A book for playing mathematics

Diego Sandoval

For Jonas and Matildė, who taught me to be here and now, filled me with their curiosity and gave my life a surge of purpose, ideas, and creative energy.

Table of contents

Preface

More Math! More Math! This book is my response to that surprising demand.

I have a very inquisitive child whose favorite word is "why." To him, an answer is never an ending, but an invitation to ask more questions. I often find myself pondering deep philosophical questions or facing improbable situations, such as drawing a blank in front of the whiteboard while he jumps on the couch demanding "More Math!"

That's how these "impromptu lectures" came to be. They follow a loose order and progression but don't make up a curriculum. They are just short explorations of topics that caught our interest at a given moment in time and were well-received, eliciting interest and, of course, more "whys."

While I draw inspiration from great educators and mathematicians like Seymour Papert and George Pólya, I don't adhere to any particular educational philosophy or method. You are free to define your own goals or let them emerge during the exploration process. In my case, I realized that the main goal was to play, discover and spend quality time together.

I don't just focus on acquiring skills or accumulating knowledge, though I do think both matter. Mathematics, music, games, and an awareness of the world around you are at the top of my list of things young minds should be exposed to. Rather than the clichéd "learning by playing," I prefer "playing for the sake of playing" — learning emerges naturally from that, and I am convinced our role is not to teach, but to facilitate and provide the right environment.

Inspired by how my kids watch the same film over and over again without losing their awe and amazement, I designed these lectures to be revisited multiple times. Feel free to skip or add content, or cherry-pick topics from here and there. I included concepts and ideas that traditional curricula consider "advanced." You might be surprised. Intuition is more powerful than convention.

> **Parent/educator Tip**
>
> I will use boxes like this one to provide tips for the teachers. By "teacher," I mean the person providing guidance or *translating* the material. Similarly, when I say "your kids," I am referring to anyone you are playing math with: your own kids, students, grandchildren, friends, younger siblings, etc.

This book is open source and available on GitHub[1] — because it's meant to grow and evolve. I welcome your feedback, suggestions, and contributions.

Mathematics is not a purely abstract endeavor, but also something you can get a tactile feel for. In that sense, this is a hands-on book—a book for scribbling, coloring, and drawing.

I hope you find this book useful the next time someone demands, "More math!"

[1]https://github.com/twaclaw/yo-yo-math

Supporting and Contributing

If you like this book or find it useful, please consider supporting it by:

- Buying me a coffee[2] ☕ (or two, three, five, seven, etc.)
- Giving a star ⭐ and contributing on GitHub[3].
- Sharing with others 💌.

[2] https://buymeacoffee.com/twaclaw
[3] https://github.com/twaclaw/yo-yo-math

Acknowledgments

I would like to express my warm thanks to everyone who has contributed to this book in one way or another.

First, my children, for asking so many questions. My wife, Gabrielė, for her thorough revisions, detailed feedback, and artistic guidance; for giving me time to write and code; and for filling our home with music and love.

Thanks also to Gerardo Restrepo for drawing the beautiful cover, and to Hamilton Carrillo and Eleonora Stassi for reading an early draft of the book and offering valuable feedback and corrections.

We often take it for granted, but the world is a better place thanks to the amazing work and dedication of the people who create open-source content and software, from Wikipedia to Blender to LaTeX. I wrote this book using Quarto and used Python and TikZ to generate the figures. I also used LDraw and Blender to create rendering of the LEGO blocks.

My friend Fabio Stassi, who has written several beautiful books, once told me that literature is always secondhand. That's why I also want to thank all the wonderful authors who have inspired me through their writing and ideas.

Thank you for reading this book and providing feedback. We hope you enjoy it as much as we enjoyed creating and playing with it.

1 Counting

1.1 Introduction

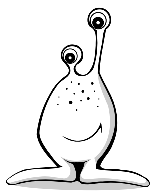
> God made the integers, all else is the work of man. *Leopold Kroneker*

> In conveying messages within the human brain, counting is far more fundamental than language. *Ronald Calinger, A Contextual History of Mathematics.*

Counting is arguably the most fundamental operation in mathematics. This seemingly trivial process provides the foundation for many areas of mathematics and is central to numerous results, algorithms, and proofs. Counting is pervasive. You will encounter it in instances ranging from playing Snakes and Ladders to solving a Rubik's Cube efficiently to proving the most elusive theorem in the history of mathematics.

Do aliens count?

Some people speculate that if we were to encounter extraterrestrial intelligence elsewhere in our universe, one thing we might have in common is the ability to count, albeit in different ways. For example, you would count differently if you had six fingers instead of ten, or no fingers at all.



1.2 Decimal numerals

The numbers we use for counting —zero, one, two, three, ...— are called counting or natural numbers and are represented with this nicely looking $\mathbb{N}$.

$$\mathbb{N} = \{0, 1, 2, 3, 4 \cdots\}$$

The three dots, ⋯, mean that the sequence continues forever, until the heavens of infinity. Whether or not to include zero in the set of natural numbers is a matter of convention—or rather, a matter of contention.

Numbers are abstract entities, so defining them is difficult. I'm not going to do that here, but you could try asking your parents or teachers what numbers are. Ask them something like, "What is the number five?" and observe their reaction 🤷.

Parent/educator Tip

What is number 5? What is a number? To delve deeper into this and other intricate philosophical questions, I recommend reading a book on the philosophy of mathematics[a]. This doesn't necessarily mean that you will find a satisfying answer, or any answer at all.

[a]For instance, "Lectures On the Philosophy of Mathematics" by Joel Davis Hamkins (MIT Press, 2020)

We represent numbers with symbols called "numerals." There are different types of numerals; we will look at some of them later.

Most of the time, we will use Hindu-Arabic numerals. These are the symbols we see everywhere and are most familiar with. We see them on our house doors, watches, phones, football jerseys, birthday candles, and the page numbers of this book.

0 1 2 3

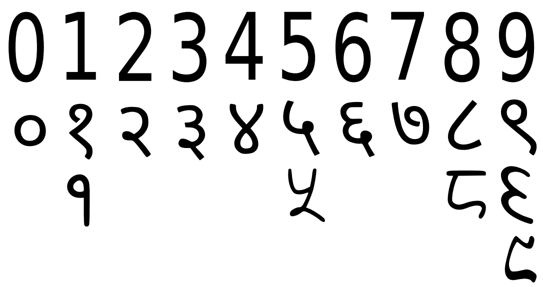
4 5 6 7

8 9

Figure 1.1: The decimal numerals

Numbers and stars

These symbols are called Hindu-Arabic numerals. Arab scholars adopted them from India. Along with the names of the stars and many other scientific and mathematical discoveries, they are among the beautiful things that the Arab tradition has bridged from other cultures or created for us.



Hindustani numerals. Source image: Wikipedia.

Go ahead and practice your strokes while enjoying the roundness of the numbers.

0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7	7	7	7	7	7	7
8	8	8	8	8	8	8
9	9	9	9	9	9	9

1.3 Counting objects

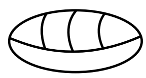
We use natural numbers to represent the quantity of objects or entities. Natural numbers are used for counting and answering questions that imply "How many?"

- How many siblings do you have?
- How many legs do dogs have?
- How many horns did a triceratops have? What about unicorns?
- How many candles were in your last birthday cake?

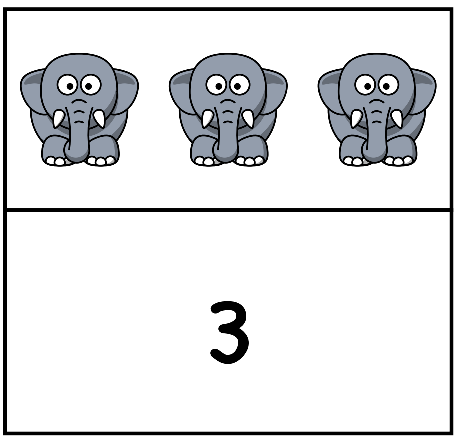
Sometimes the answer is a very special quantity: zero, 0. Consider the answer to the question "how many horns do you have?" Hopefully none.

A number to represent nothingness

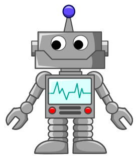
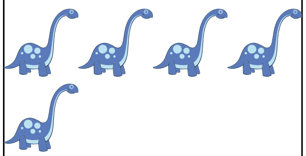
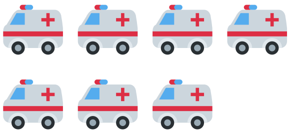
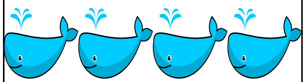
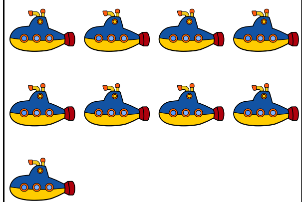
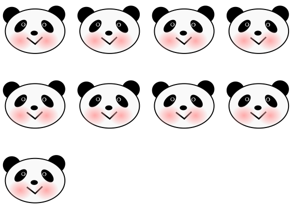
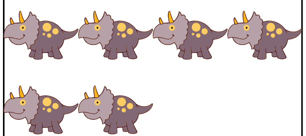
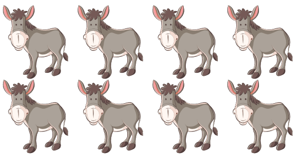
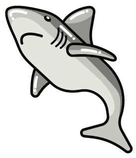
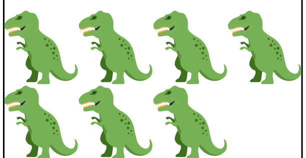
The invention of zero (0), a number representing nothingness, was a significant milestone in the history of mathematics. Not all cultures recognized the concept of zero immediately or had a value for representing nothingness. The one in the picture is the Mayan zero. Doesn't it look like a rugby ball? Source image: Wikipedia.

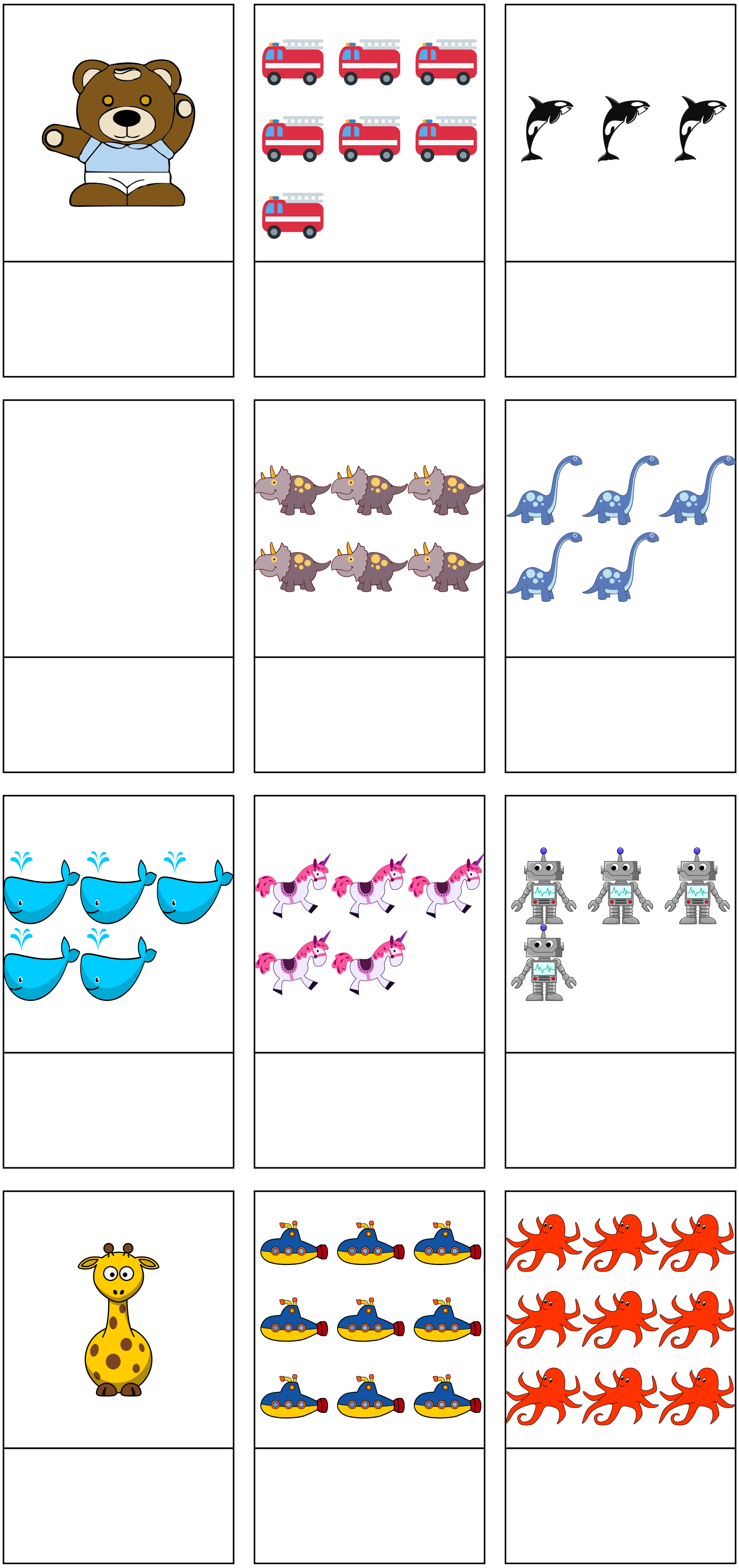


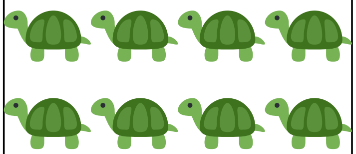
Let's see how to use decimal numerals to represent quantities. For example, how many elephants do you see in the next figure? You would agree that the answer is three.

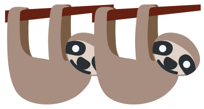


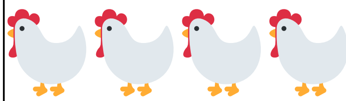
Let's practice our counting skills!

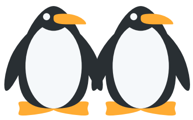
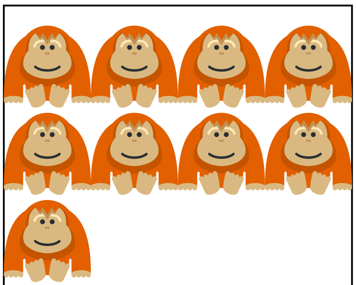
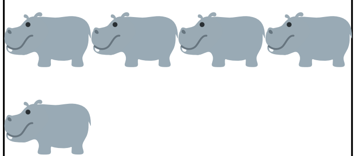
		
0	1	5
		
		
		

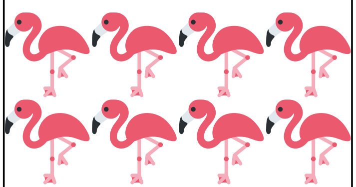








Bedtime stories drawn in the sky

Orion, the son of Poseidon and Euryale, was an unrivaled hunter. Even Artemis, the goddess of the hunt, enjoyed hunting with him. However, her brother Apollo grew jealous of Orion and sent a giant scorpion to sting him. After Orion died, Artemis forgave Apollo, and they immortalized him in the sky as the eponymous constellation. He is followed by his two faithful dogs, represented by the constellations Canis Major 🐕 and Canis Minor 🐕. The dogs are pursuing the poor hare, represented by the constellation Lepus 🐇. Scorpius, a summer constellation, is usually not seen with Orion at the same time.

You can find more about this story in D'Aulaires Book of Greek Myths. Doubleday, 1962, and Penguin Random House 1992.

This hunting scene is depicted in the night sky (just look up), as shown in figure Figure 1.2. The neighboring constellations Taurus ♉, the bull, and Gemini ♊, the twins, are also shown in the figure.

Answer the following questions based on figure Figure 1.2:

- How many stars are in the Orion Belt? Do you know their names?
- How many stars can you see in the Greater Dog (Canis Major)? How many do you see in the smaller one, Canis Minor? I know the Lesser Dog doesn't really look like a dog.
- How many stars are there in Orion's lion skin on his left arm? Sometimes, instead of a lion skin, it is depicted as a shield, but the number of stars remains the same.
- How many orange-red stars do you see in this picture? In which constellations? Do you know their names?

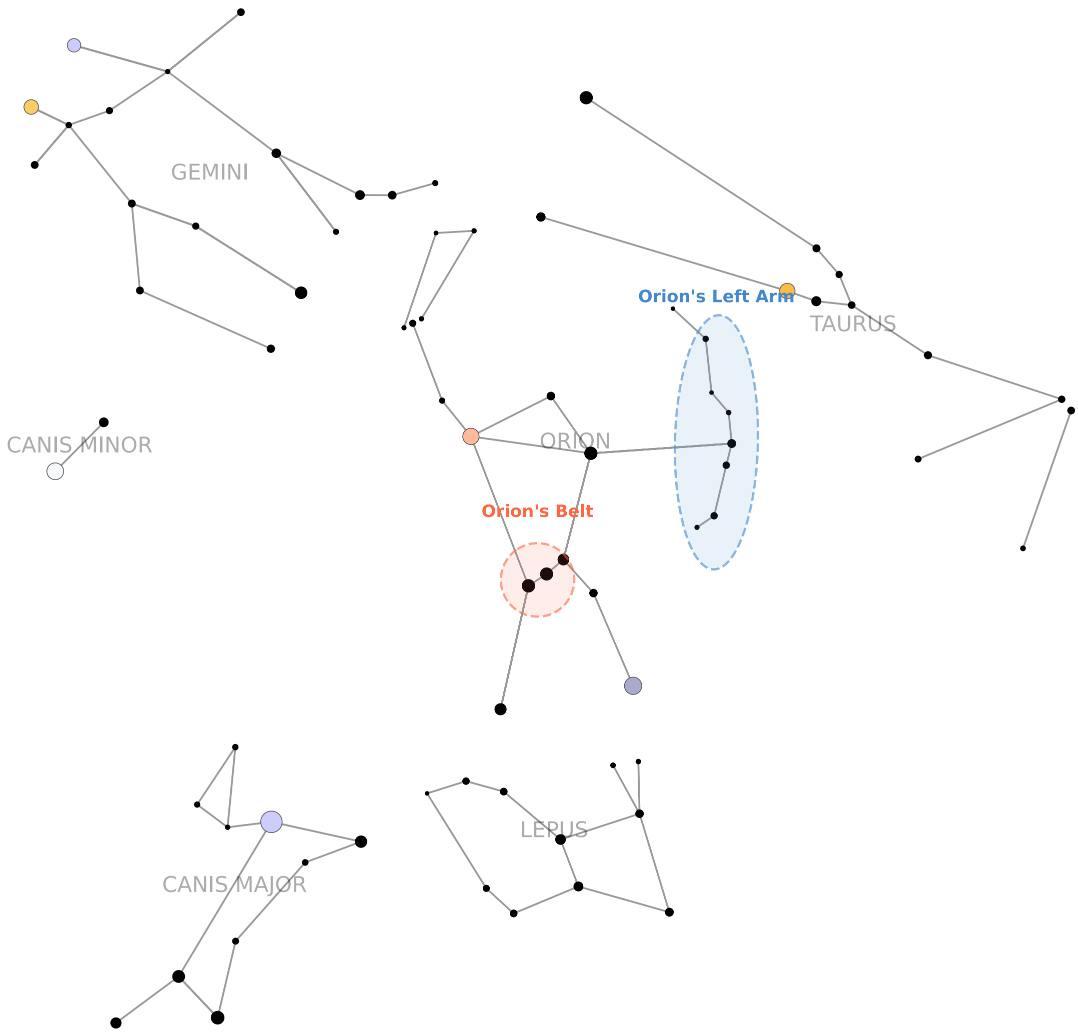


Figure 1.2: Winter night sky.

Parent/educator Tip

If this example sparked your interest in astronomy and the night sky, here are a series of recommendations:

- Stellarium[a]: A free, open-source planetarium software.
- Constellations (Princeton University Press, 2012) by Govert Schilling: A beautifully illustrated book about constellations, their history and mythology.
- Canis Major[b]: A project combining the two. It provides animations, including constellations, groups of constellations, orbits, etc.

[a] https://stellarium.org/
[b] https://github.com/twaclaw/canismajor

1.4 Addition and substraction

Imagine that you are building a tower out of LEGO bricks. Each time you add a brick to the top, the total number of bricks in the tower increases, meaning the tower gets taller. If you start with one brick and add a second, the tower is now one brick taller. Now, you have 2 bricks.

(a) Adding 1 block to the tower. (b) Tower is now 2 blocks tall.

Figure 1.3: Adding 1 block.

Similarly, if you start with 2 blocks and add 3, your tower's height increases by 3, it is now 5 blocks tall.

Adding up means increasing a quantity by another quantity, or combining two quantities to obtain a new, larger, quantity. There is small nuance to this definition that we will review once we have introduced negative numbers.

We use the plus symbol $+$ to represent addition, and the symbol $=$, or equals, to represent the result.

Using this notation we can represent the previous statements more compactly as:

- 1 plus 1 equals 2 $\rightarrow$ $1+1=2$, and
- 3 plus 2 equals 5 $\rightarrow$ $3+2=5$.

(a) Adding 3 blocks to the tower. (b) Tower is now 5 blocks tall.

Figure 1.4: Adding 3 blocks.

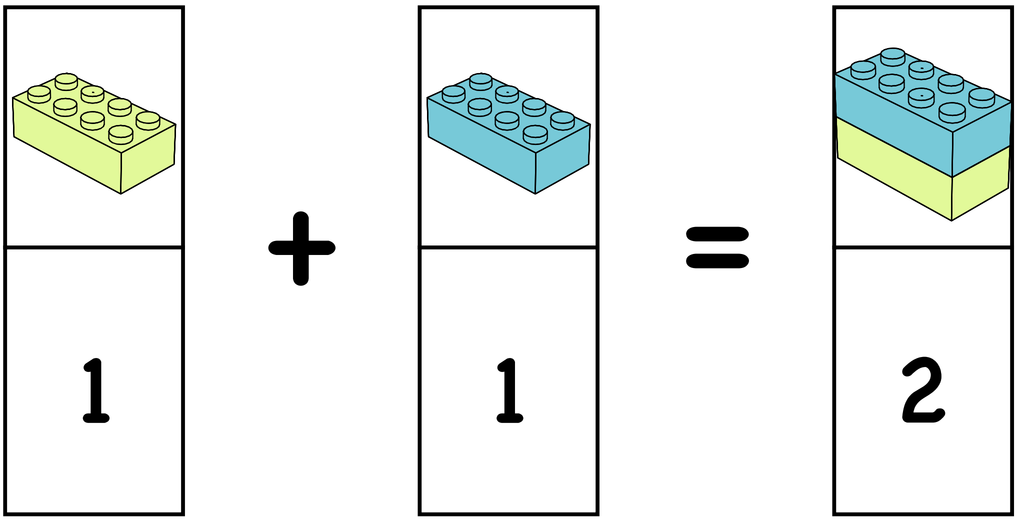


Figure 1.5: 1 LEGO block plus 1 LEGO block equals 2 LEGO blocks.

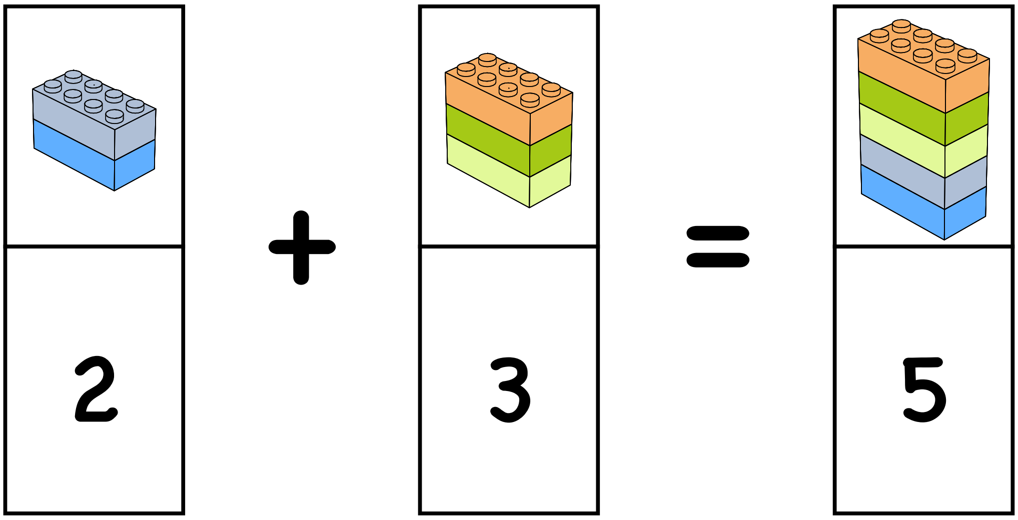


Figure 1.6: 2 LEGO blocks plus 3 LEGO blocks equals 5 LEGO blocks.

When you're done playing or you want to build something more exciting than a tower, or when your mom asks you to clean up, you start removing blocks out of the tower and putting them in the LEGO box. This decreases the number of blocks in the tower, making the tower shorter and shorter.

Substraction means **removing** things. We use the *minus* sign, $-$ to represent this. For instance, if your LEGO tower has 5 blocks and you *remove* 3, you are left now with 5 minus 3, which equals 2. Using the new notation we just learned: $5-3=2$.

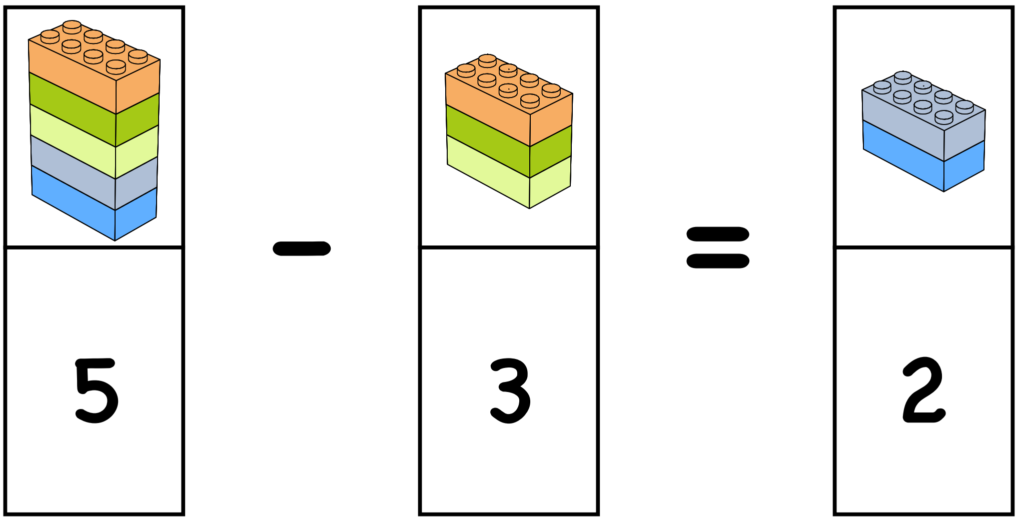
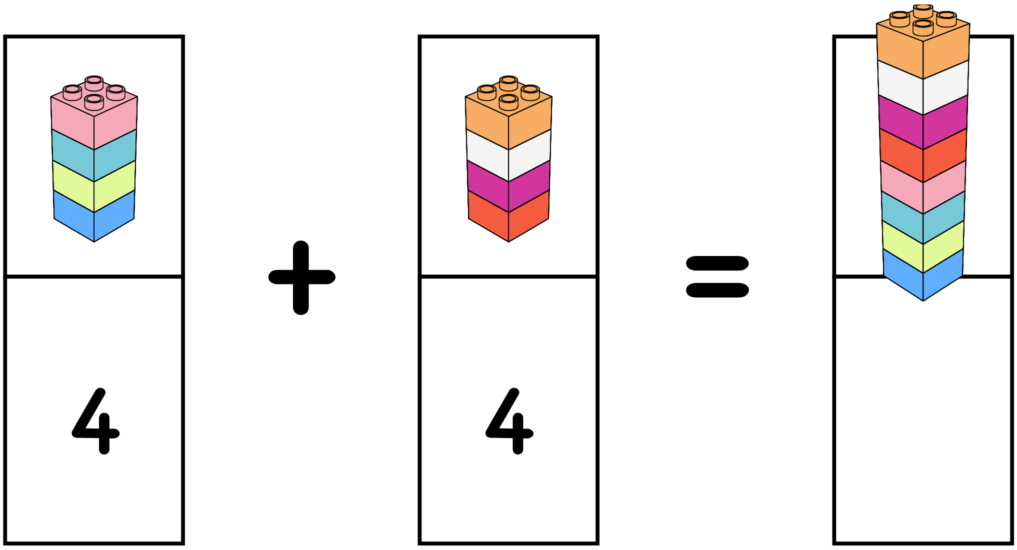
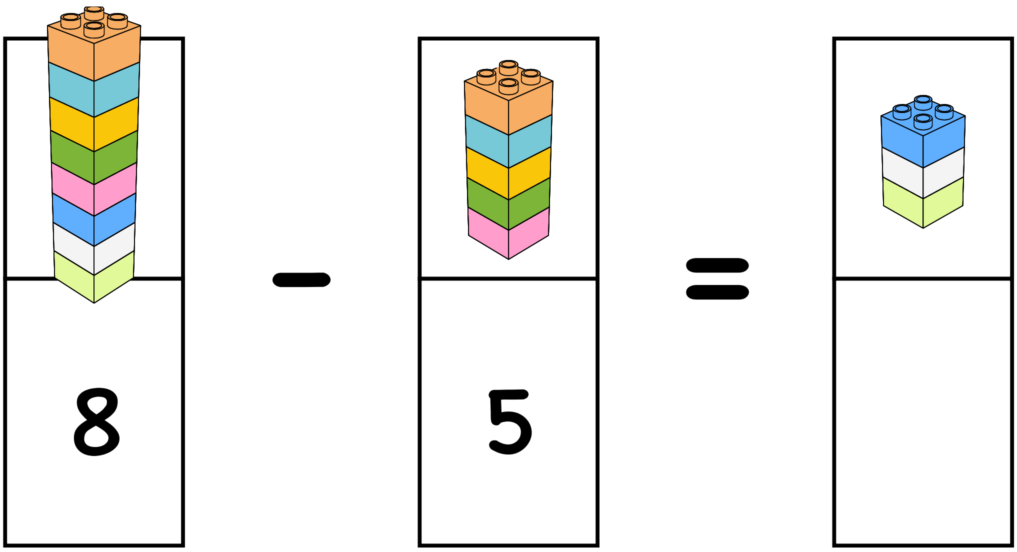


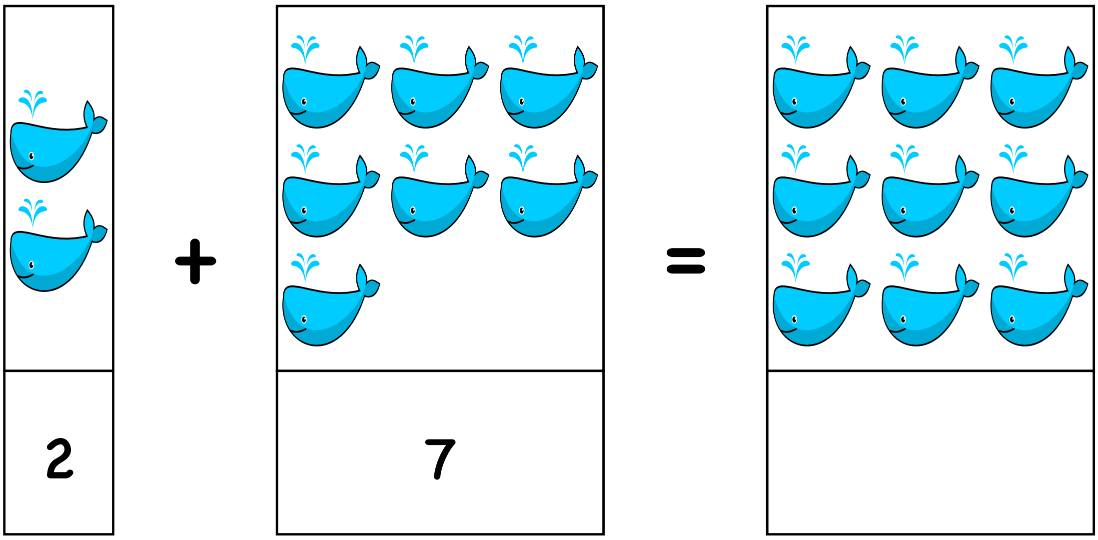
Figure 1.7: 5 LEGO blocks minus 3 LEGO blocks equals 2 LEGO blocks.

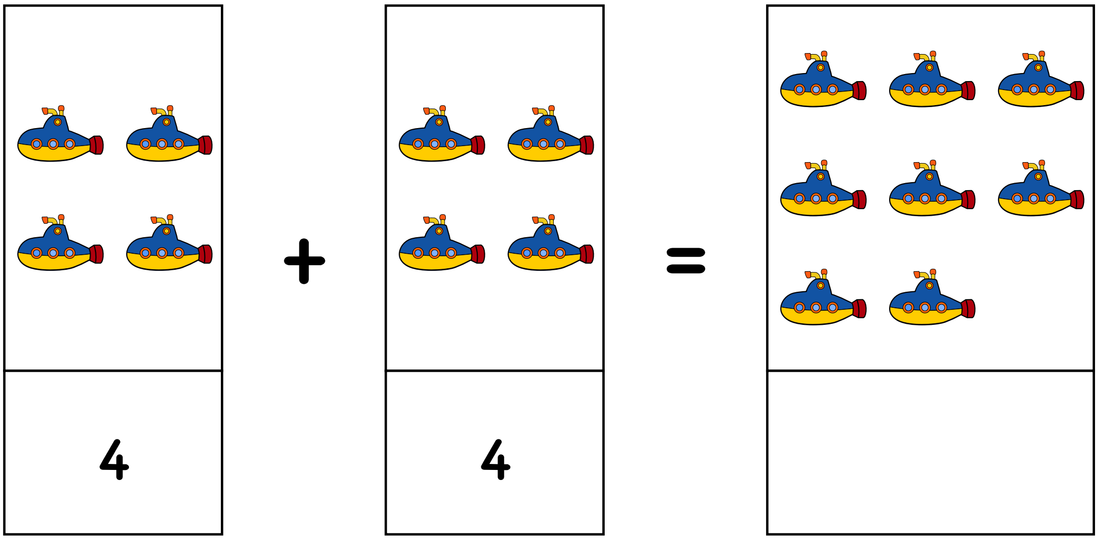
Exercises

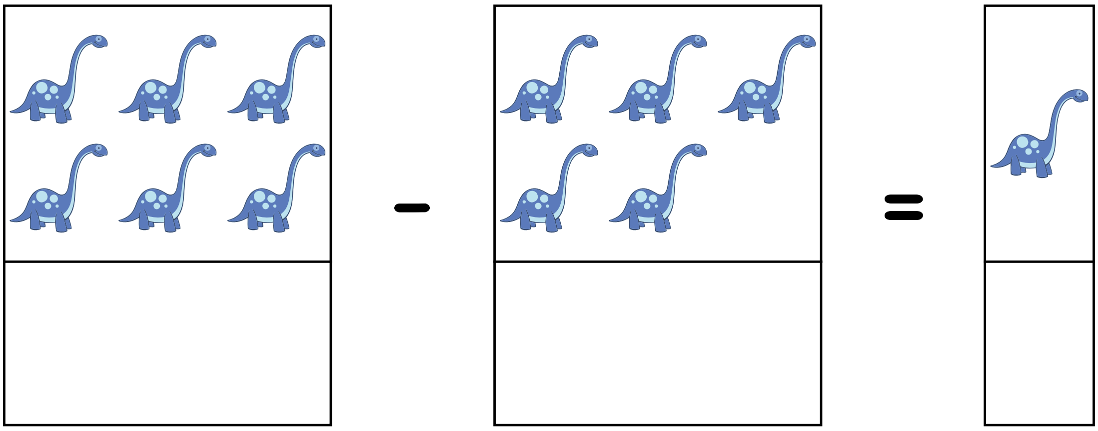


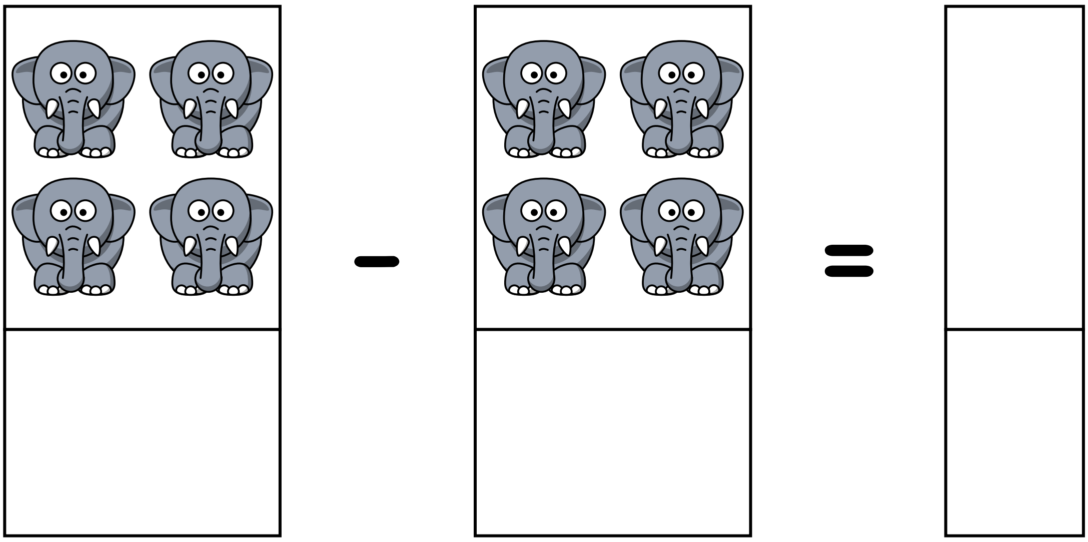


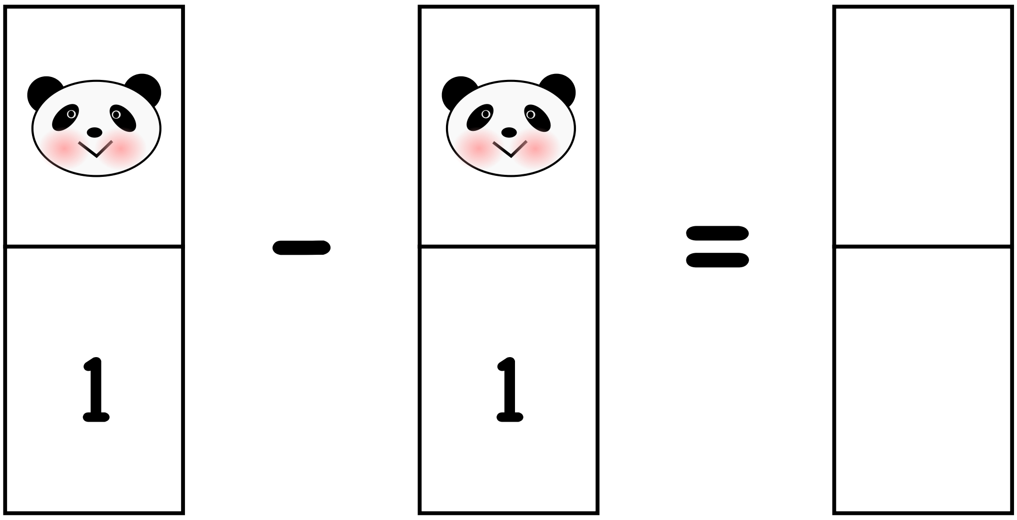
Of course, you can add and substract with other things than LEGO blocks.


+
=
2
7


+
=
4
4

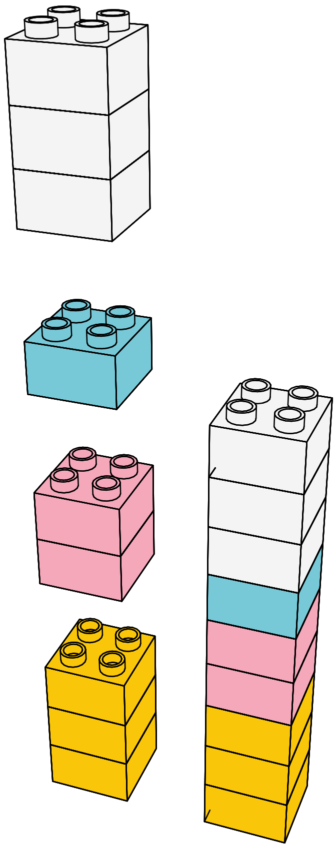

−
=


−
=

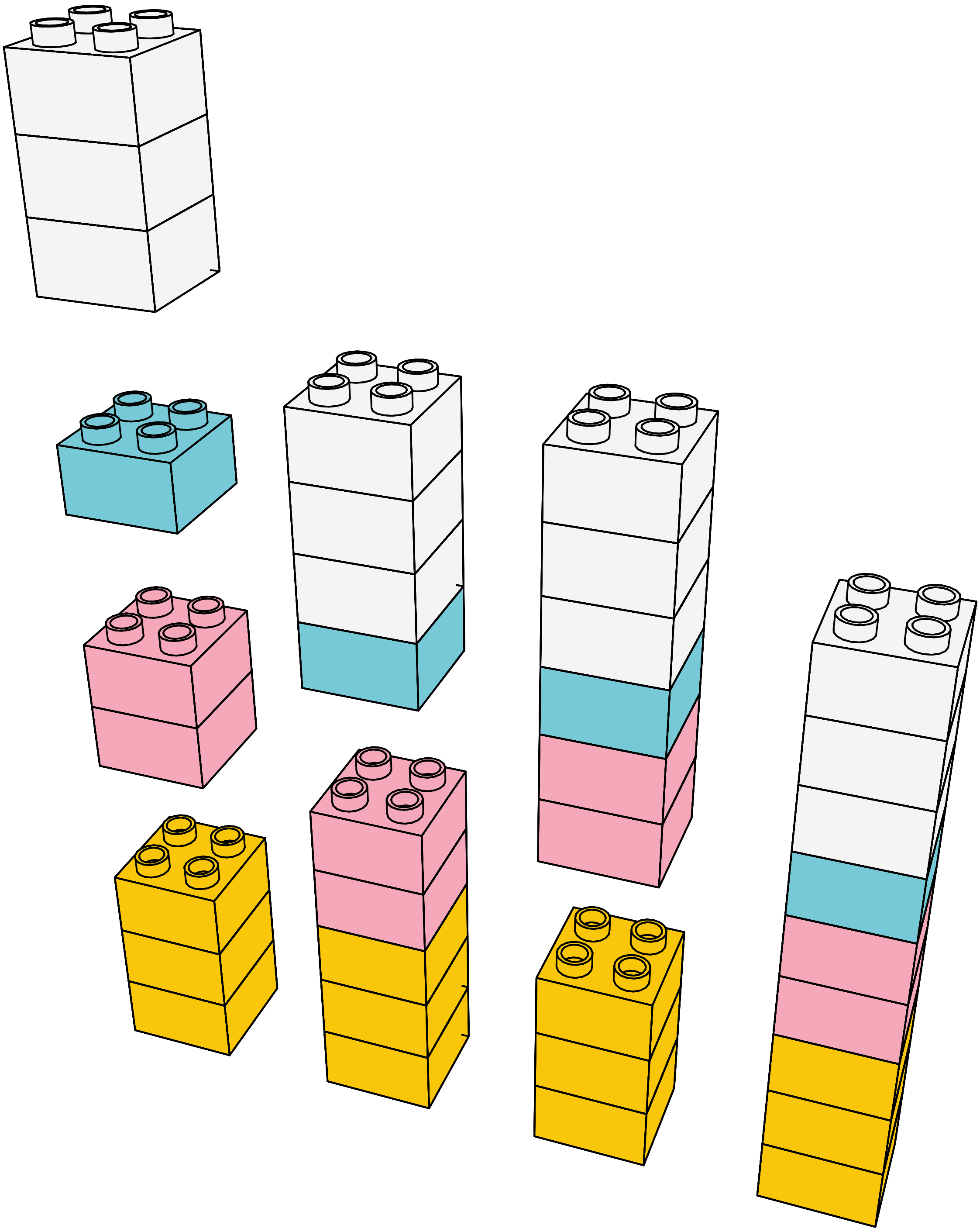

1
−
1
=

Addition means combining quantities. So far, we have combined two quantities, but we can combine more.

For example, in the following figure, we are adding 3 white bricks, 1 blue brick, 2 pink bricks, and 3 yellow ones. This represents the sum $3 + 1 + 2 + 3$, which equals 9.



When adding more than two numbers, it is helpful to break the process down into smaller steps. We can group the summands in different ways. The cool thing is that the result will always be the same, no matter how we group them. This property of addition is called **associativity**. For example, in the following figure, we are grouping the bricks in different ways, but we always end up with 9 bricks.



We can use parentheses to indicate which numbers we should group together and add up first.

$$\underbrace{\underbrace{(\underbrace{(3+1)}_{3+1=4}+2)}_{4+2=6}+3}_{6+3=9} = 9$$

However, we can also associate numbers differently:

$$\underbrace{\left(\underbrace{(3+1)}_{3+1=4}+\underbrace{(2+3)}_{2+3=5}\right)}_{4+5=9} = 9$$

Now that you have that knowledge, let's solve the following sums. You can use beads, your fingers, or LEGO bricks to help with the calculations.

$$3+2=\boxed{5}$$

$$1+1=\boxed{2}$$

$$2+3+1=\boxed{}$$

$$1 + 0 = \square$$

$$0 + 1 + 2 + 3 = \square$$

$$9 - 0 = \square$$

$$9 - 1 = \square$$

$$9 - 2 = \square$$

$$9 - 3 = \square$$

$$7 - 4 = \square$$

$$0 + 0 = \square$$

$$3 - 3 = \square$$

$$3 + 5 = \square$$

$$5 + 3 = \square$$

$$5 + 3 - 3 = \square$$

$$5 + 2 - 3 = \square$$

The order doesn't matter

Sometimes, the order of things matters. For example, it matters whether you put on first your shoes  or your socks .
Other times, order doesn't matter, such as when adding up numbers. For instance, did you notice that $3 + 2$ yields the same result as $2 + 3$?

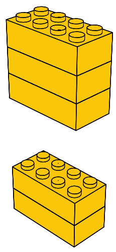


Figure 1.8: $2 + 3$

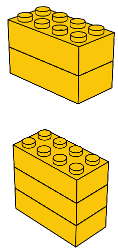


Figure 1.9: $3 + 2$

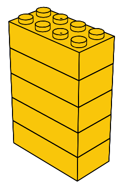


Figure 1.10: $2+3 = 3+2 = 5$

In the general case we have

$$a + b = b + a$$

a and b represent any two numbers. We say that addition is **commutative**, meaning the order of the numbers being added does not matter.

Provided you enter the right answers in the previous examples, we say that the equality holds. Try now yourself to create some equalities that hold, both with additions ('+') and substractions ('−').

Parent/educator Tip

Encourage the kids to do this on their own, you can provide a few examples.

$$\square \qquad \square \qquad \square = \square$$

$$\square \qquad \square \qquad \square = \square$$

$$\square \qquad \square \qquad \square = \square$$

$$\square \qquad \square \qquad \square = \square$$

$$\square \qquad \square \qquad \square = \square$$

$$\square \qquad \square \qquad \square = \square$$

$$\square \qquad \square \qquad \square = \square$$

$$\square \qquad \square \qquad \square = \square$$

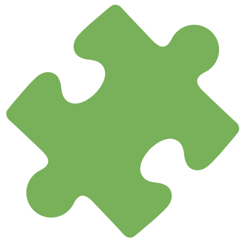
$$\square \qquad \square \qquad \square = \square$$

$$\square \qquad \square \qquad \square = \square$$

1.5 Playing hide-and-seek with numbers

Playing hide-and-seek

Did you know that with the operations you just learned, you can actually play hide-and-seek? The game consist in finding numbers that are hiding? It is like finding the missing piece of a puzzle.



In elementary algebra we call those hiding numbers "unknowns" or "variables".

To find the value of the hidden number in the following equation, ask yourself: What number plus 2 equals 7?

 $+ 2 = 7$

The answer:

 $= 5$

Because

$$5 + 2 = 7$$

Exercises

In the following equations, find the uknown. The hidden numbers are hidding inside a box like this one $\square$, go and find them!

$$\square + 3 = 8$$

$$\square + 7 = 8$$

$$7 + \square = 8$$

$$2 - \square = 0$$

$$2 - \square = 1$$

$$\square + 3 = 8$$

$$7 + \square = 7$$

$$7 - \square = 7$$

$$9 + \square = 9$$

$$9 - \square = 9$$

$$1 + \square = 2$$

$$2 - \square = 1$$

Fun Fact

In algebra those unknown variables are usually represented by letters, like x or α.
So, instead of $3 + \square = 7$, you will see $3 + \alpha = 7$. We will learn later on powerful methods for finding not one but multiple of those sneaky numbers, provided they exist!

In many instances in science and engineering you are not given these equations, but instead some observations or a problem. Nature speaks mathematics, but still you will have to convert observations into equations.

- Example 1: If I tell you that Zoe is 2 years older than you, and then I ask you what's Zoe's age? How would you go about and solve this problem? In this kind of problems you always have two categories, what you know: the data, observations, given information, and what you don't know, the sneaky unknowns. In this case you know your own age, let's say 5, and you know that Zoe is 2 years older. The resulting equation would be:

$$5 + 2 = \square$$

- Example 2: the temperature tomorrow is going to be 3 degrees colder than today. It is going to fall to 6 degrees. What is the temperature today? In this case the data is the temperature tomorrow, 6 degrees, and the relationship between today and tomorrow, that is the temperature tomorrow is 3 degrees colder than today. The unknown is the temperature today. The resulting equation would be:

$$\square - 3 = 6$$

- In the Zoo there are 2 lions 🦁🦁, 2 elephants 🐘🐘, 1 giraffe 🦒 and an unkown number of orangutans 🦧?. If the total number of animals in the zoo is 9, how many orangutans are there?

$$\square \quad + \quad \square \quad + \quad \square = \square$$

$$\square \quad - \quad \square = \square$$

- If I remove all the red blocks from a tower that is 7 blocks tall, I am left with 2 blocks. How many red blocks were there in the tower?

$$\square \quad \square \quad \square = \square$$

- The distance between two cities is 7 kilometers. If I have already traveled 4 kilometers, how many kilometers do I have left to travel?

$$\square \quad \square \quad \square = \square$$

1.6 A preview of multiplication

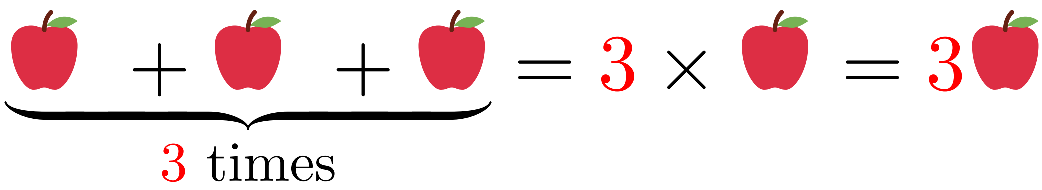
A special case of addition (or substraction) is when you add a number to itself. For instance $3 + 3$, what we can read as three plus three, but also two *times* three. There is nothing stopping us at two times, so we can also continue to $3 + 3 + 3$, which is read three *times* three, or $3 + 3 + 3 + 3$ or four *times* three.

This operation, repeated addition, is called multiplication and is represented by the *times* symbol $\times$.

We can then write $3+3+3+3$ in a more compact way as 4×3 that is read 4 *times* 3.

$$\underbrace{3+3+3+3}_{\boxed{4\ \text{times}}} = 4 \times 3$$

Here is another example: 1 apple + 1 apple + 1 apple is the same as 3 times 1 apple, which is 3 apples.



More generally, if a number a is added to itself n times

$$\underbrace{a+a+\cdots+a}_{\boxed{n\ \text{times}}} = n \times a$$

We will revisit multiplication later on. For the time being, you can already do multiplications, albeit in a slower way, using one of your superpowers: addition.

Exercises

Do the following multiplications by doing adding up repeatedly.

$$1 \times 2 = 2 \qquad = \boxed{2}$$

$$2 \times 2 = 2 + 2 \qquad = \boxed{4}$$

$$3 \times 2 = 2 + 2 + 2 \qquad = \boxed{}$$

$$4 \times 2 = 2 + 2 + 2 + 2 \qquad = \boxed{}$$

$$2 \times 3 = 3 + 3 \qquad = \boxed{}$$

$$3 \times 3 = 3 + 3 + 3 \qquad = \boxed{}$$

$$2 \times 4 = 4 + 4 \qquad = \boxed{}$$

1.7 Comparing quantities

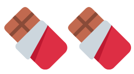
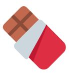
- What would you prefer, two ice-creams or one ice-cream?
- Are your siblings older or younger than you?
- What about your best friends, are they older than you?
- What do you prefer 2, 1 or 0 portions of onion soup?

> **Parent/educator Tip**
>
> Encourage the kids to reason about their answers. Ask many "Why's?"

Think for a second how you answered those questions. What you did, even if unconsciuously, was a comparison between two or more quantities.

We already saw a kind of relationship between two quantities, namely when they are *equal*, now we are going to see the two other options.

Name	Symbol	Example	Description
Equal	$=$	$3 = 3$	3 is equal to 3
Greater-than	$>$	$3 > 1$	3 is greater than 1
Less-than	$<$	$0 < 1$	0 is less than 1

For instance I am sure you would prefer 2  bars of chocolate instead of 1  because $2 > 1$ and 0 portions of cauliflower soup instead of 1 because $1 > 0$.

> **Fun Fact**
>
> In a way, the greater-than symbol $>$ is the same as the less-than one $<$. What you have to remember is that the open part always points to the larger quantity, and the angle to the smaller one.
> For instance, if you see something like $7 > 2$ you can read it as 7 is greater than 2, but also as 2 is less than 7, and write it as $2 < 7$. Same thing!

Time to practice.

Exercises

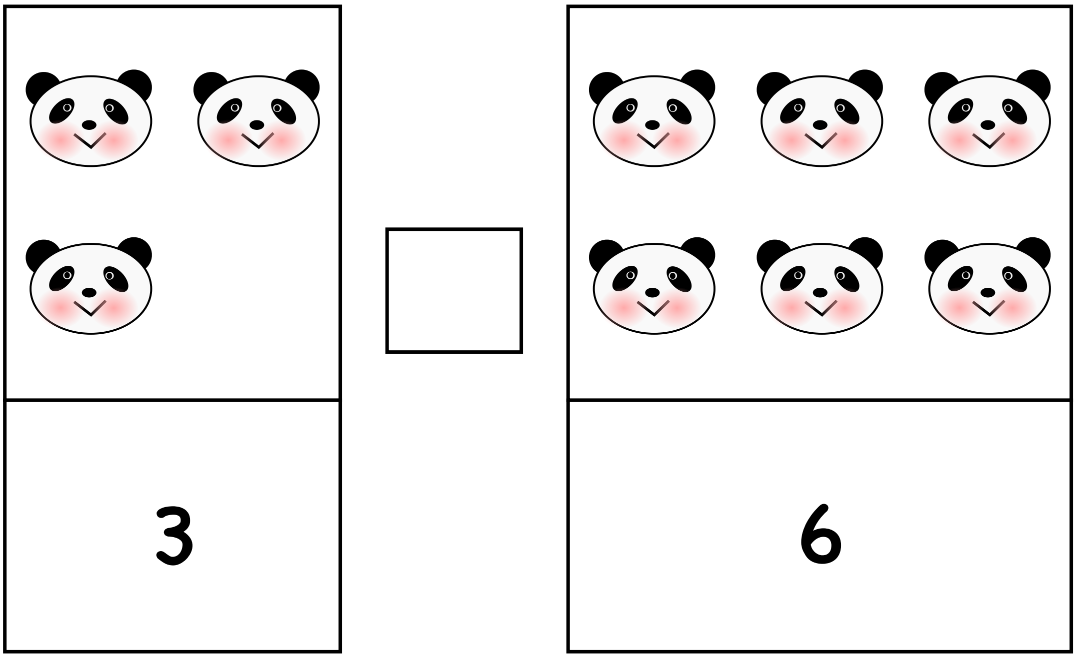


Figure 1.11: Write down the correct symbol ($>$ or $<$) in the box.

$$3 \quad \boxed{>} \quad 0$$

$$3 \quad \boxed{<} \quad 4$$

$$3 \quad \boxed{} \quad 1$$

$$3 \quad \boxed{} \quad 3$$

$$9 \quad \boxed{} \quad 8$$

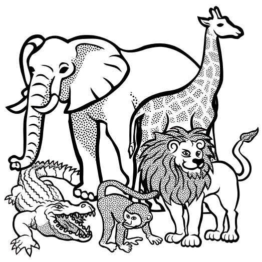
$$7 \quad \boxed{} \quad 7$$

$7 \quad \square \quad 8$

$7 \quad \square \quad 9$

$1 \quad \square \quad 0$

Optimization



Which animal is the tallest? Which one is the heaviest? These kind of questions implies also comparisons of quantities, in this case heights and weights which are numbers. Given some quantities like $2, 4, 7, 2, 8, 9, 2$, we can define:

- **Minimum**: the smallest number. The number that is less-than ($<$) all other numbers in the collection. A number x^* such that $x^* \leq y$ for all y in the collection. There might multiple instances of this number. In this example, the minimum value is 2.
- **Maximum**: the largest number. The number that is greater-than ($>$) all other numbers in the collection. A number x^* such that $x^* \geq y$ for all y in the collection. There might be also multiple instances of this number. In this example, the maximum is 9.

These points (*minima* and *maxima*) are called **extrema** (plural of extremum) and they are very important in many branches of mathematics, physics, engineering and economics.

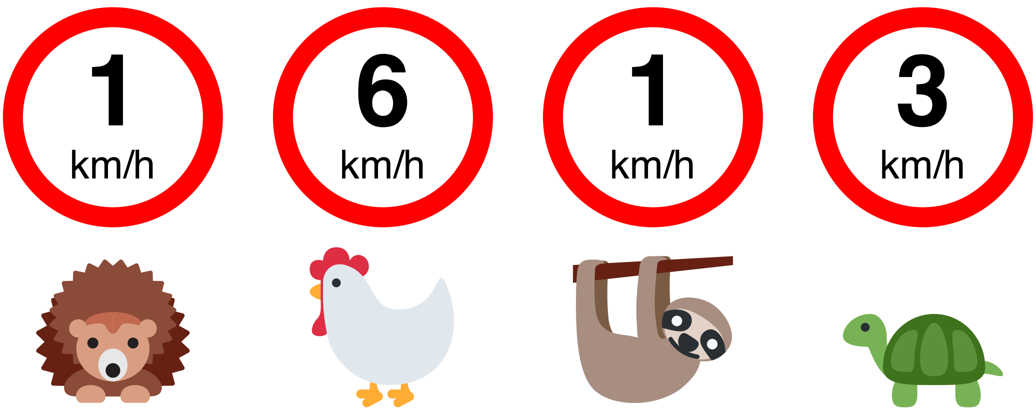
Imagine you are going to have a birthday party and the following kids are attending.

Name	Age	Name	Age
Jonas	5	Miguel	4
Lea	7	Zoe	4
Matilde	2	Amelie	6
Lucas	2	Zelia	9

Answer the following questions:

- Who is older, Jonas or Lea?
- Who is the oldest between Miguel, Zoe and Amelie?
- Who is the youngest kid in the party?
- Who is the oldest kid in the party?

Based on the following figures, write down the corresponding relation and answer the following questions:



- Who is faster, the rooster or the turtle?

$$\boxed{6} > \boxed{3}$$

Answer: The rooster  is faster.

- Who is faster, the hedgehog or the rooster?

$$\square \qquad \square \qquad \square$$

- Who is slower, the sloth or the turtle?

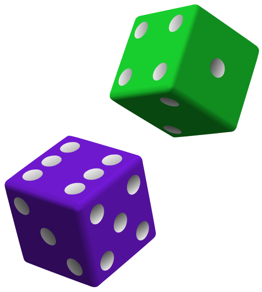
$$\square \qquad \square \qquad \square$$

- Which animal is the slowest?

$$\square \quad < \quad \square \quad < \quad \square \quad < \quad \square$$

Time to play

Game Theory

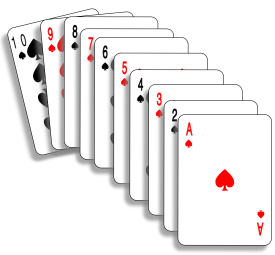


Now that you can compare quantities, you can start playing. One the simplest and oldest games is the dice game. In some languages the word for die is “bone”, because people used to play with certain animal bones. Take turns and roll a die, the player with the highest number wins. If you play with multiple dice, the player with the highest sum wins.

As we will see soon in Section 1.9, dice are usually built as regular polyhedral solids. The most common one is a cube or hexahedron (with 6 faces). But there are also dice with 4, 8, 10, 12 and 20 faces.

Figure 1.12: A 20-sided dice. Source image: Freesvg.org.

Game Theory



Card games are another instance where comparing is fundamental. In many games, the objective is to sort the cards in increasing or decreasing order.

For instance, The Mind is a cooperative card game where players have to play cards in alternate turns, and the cards have to be played in increasing order. The catch is that the players are not allowed to talk to each other, so they have to rely on their intuition and on the timing of their plays to guess when to play their cards.

1.8 Introducing the Soroban abacus

My fellow townsman, Jaime García Serrano, has a superpower. He can perform complex mathematical computations in his head in the blink of an eye. That's why people call him "the human calculator."

He claims that he is not a genius, but rather that he developed a method based on his early work with the Japanese abacus, as well as through a lot of practice.

Do you see? Like playing the cello, dancing, or climbing mountains, math is something you get better at by practicing.

Parent/educator Tip

Having a real abacus can be practical and fun. Make sure you spend some time teaching the kids how to use it and which fingers to use. There are plenty of good online resources for that.
This is a Python library for drawing these Soroban abacuses: https://github.com/twaclaw/soroban.

A Soroban abacus looks like this:

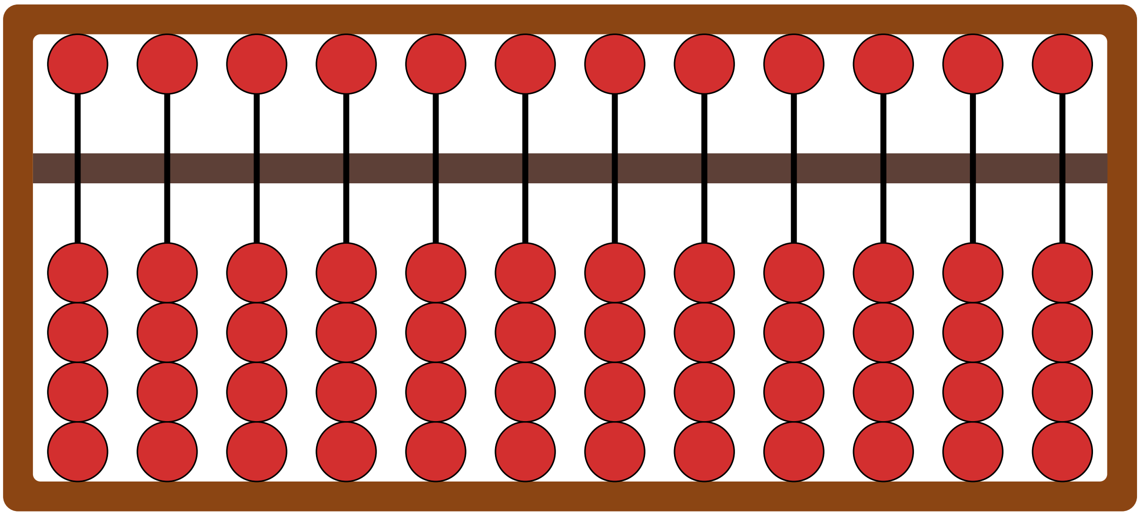


Figure 1.13: A representation of a Soroban abacus.

The number of columns can vary. For now, we will be using a tiny, cute one-column Soroban.

The beads on an abacus represent quantities, and you can perform operations like addition and subtraction with them. The dark bar inside the abacus is called the "answer" bar. Beads only add to the value if they touch the answer bar.

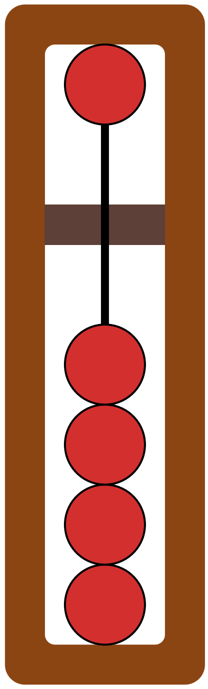


Figure 1.14: A tiny, 1-column Soroban. In this position, the beads represent 0.

The beads on this one-column abacus represent numbers from 0 to 9. The upper beads represent 0 when they are not touching the bar and 5 when they are. The lower beads each represent the value of 1.

Review the following examples, then complete the exercises.

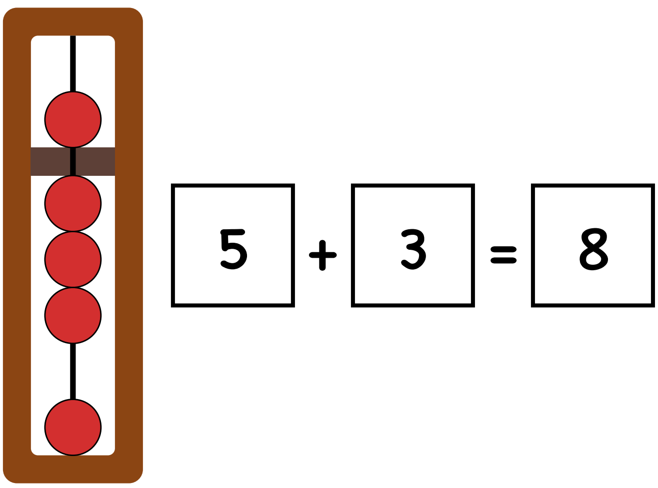
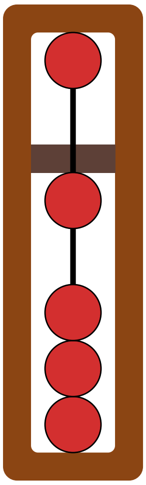
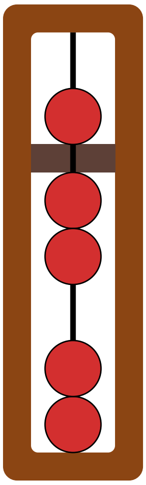


Figure 1.15: Number 8. Notice that the upper bead is touching the answer bar, which means it contributes 5 to the total value. The three lower beads are also touching the answer bar, which means they contribute 3 to the total value. In total, we have $5 + 3 = 8$.

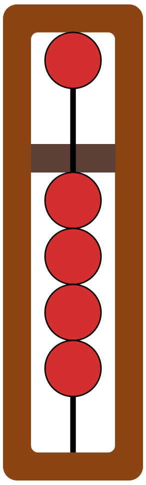
Exercises



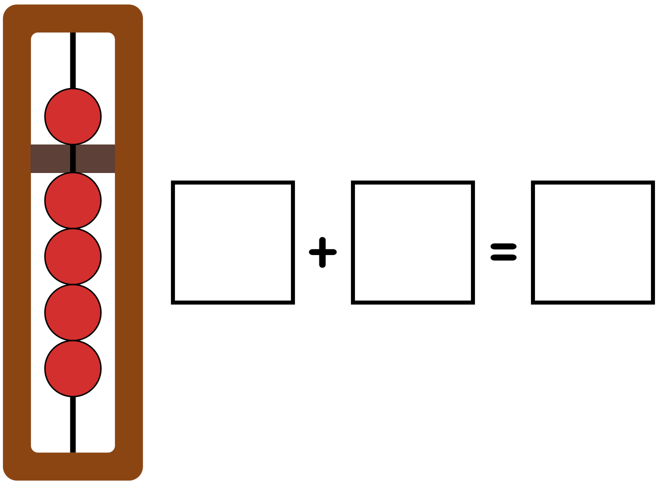
$\square + \square = \square$

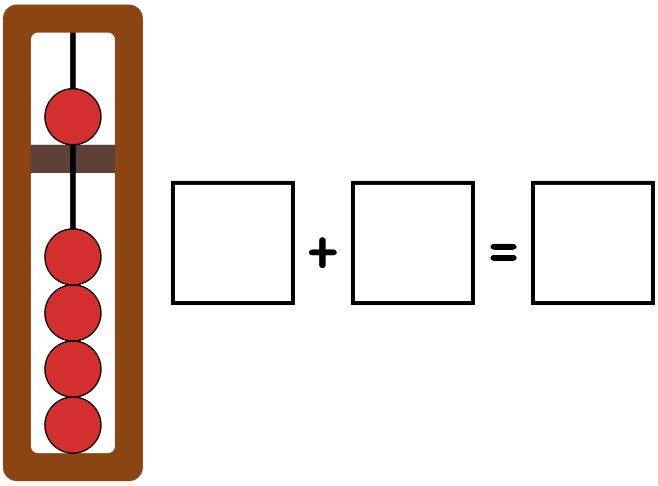


$\square + \square = \square$



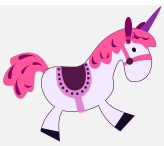
$\square + \square = \square$

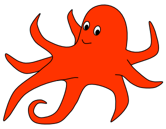




1.9 Triceratops, unicorns  and polygons

Some things are called after some inherent number or quantity.

	Name	Meaning
	**Tri**ceratops	3-horned
	**Uni**corn (**Mono**ceros)	1-horned

	Name	Meaning
	**Octo**pus	8-leged

Another prominent example of things named after a number are geometric figures.

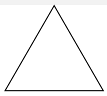
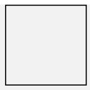
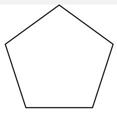
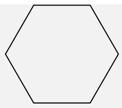
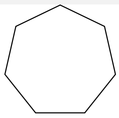
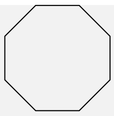
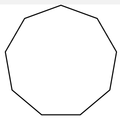
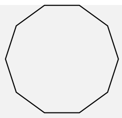
Geometric figures

Earth measurement

Geometry is an important branch of mathematics. The word "geometry" comes from the Greek word "γεωμετρία" (geometría) meaning "earth (or land) measurement."

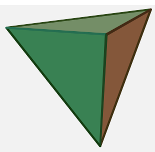
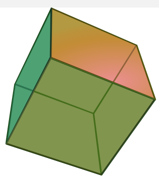
According to an unverified story, there was an inscription at the entrance of Plato's Academy that read: "ἀγεωμέτρητος μηδεὶς εἰσίτω", "Let none but geometers enter here."
You can think of these notes as your entry ticket.

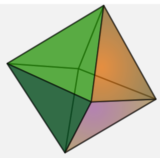
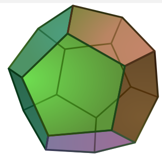
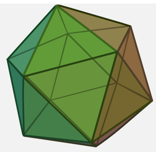
Polygons are 2-dimensional shapes. Regular polygonons are called after the number of their sides (or angles).

Number	Greek number	Polygon name	Polygon
3	τρία	**tri**angle	
4	τέσσερα	square (**tetra**gon)	
5	πέντε	**penta**gon	
6	έξι	**hexa**gon	
7	επτά	**hepta**gon	
8	οκτώ	**octa**gon	
9	εννέα	**ennea**gon	
10	δέκα	**deca**gon	

If we escape the 2-dimensional world of polygons and move to three dimensions in which we live, we can find polyhedra (meaning "many faces"). A die  or a Rubik's cube are examples of a polyhedron with 6 faces. The different polyhedra are also named after the number of their faces. The most famous ones are the Platonic solids. There are only five of them, and they are named after the number of their faces.

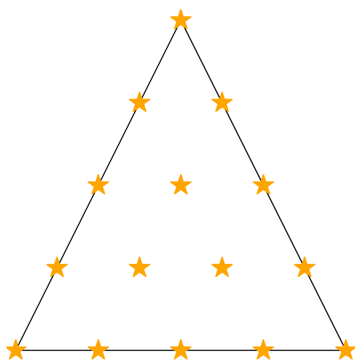
Table 1.1: The Platonic solids. Source images: Wikipedia

Number	Greek Number	Name	
4	τέσσερα	**tetra**hedron	
6	έξι	**hexa**hedron	

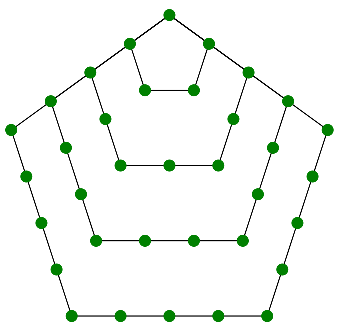
Number	Greek Number	Name	
8	οκτώ	**octa**hedron	
12	δώδεκα	**dodeca**hedron	
20	είκοσι	**icosa**hedron	

Beautiful natural numbers

Now that you know what pentagons and triangles are, you can appreciate the beauty of triangular and pentagonal numbers. In general, figural numbers (there are also square, hexagonal, etc.) are beautiful natural numbers that can be arranged in different geometric patterns. For example, the following are the fifth triangular number and the fifth pentagonal number.



$T_5 = 15$



$P_5 = 35$

To identify a triangular or pentagonal numbers, you just have to count the number of dots in them.

Parent/educator Tip

This is the Python library for generating and drawing triangular and pentagonal numbers: https://github.com/twaclaw/figural.

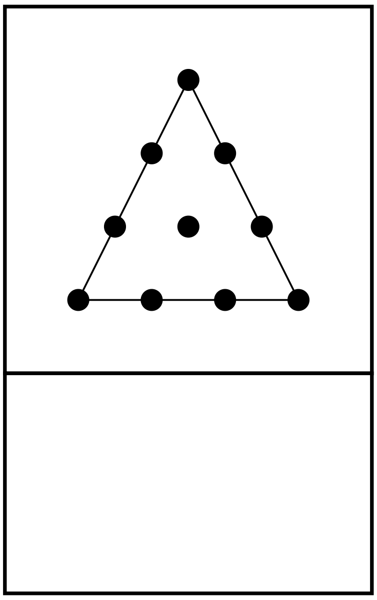


Figure 1.16: Count the number of points

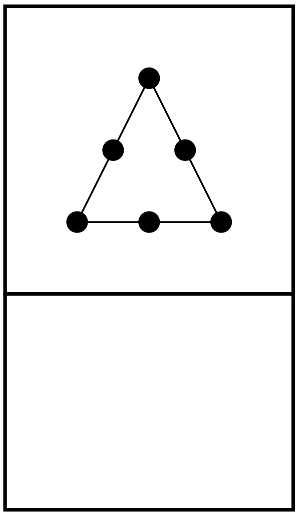


Figure 1.17: Count the number of points

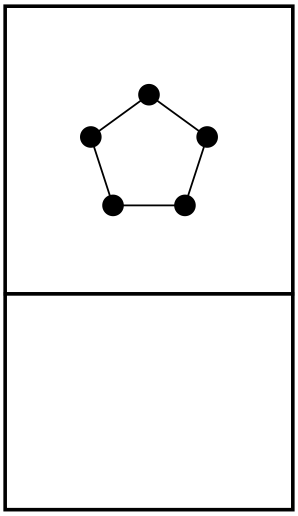


Figure 1.18: Count the number of points

1.10 Additional thoughts on addition

Pun intended 😉.

In a way, counting and adding up are the same thing. Each of the counting numbers can be obtained by adding 1 to the previous one.

$$0 \xrightarrow{+1} 1 \xrightarrow{+1} 2 \xrightarrow{+1} 3 \xrightarrow{+1} 4 \xrightarrow{+1} \cdots \xrightarrow{+1} n-1 \xrightarrow{+1} n$$

That means that if we start with 0, the next number is $0 + 1$, the next one is $1 + 1$, and so on.

$$\begin{array}{cccccc}
0 & & & & \\
1 & = & 0 & + & 1 \\
2 & = & 1 & + & 1 \\
3 & = & 2 & + & 1 \\
4 & = & 3 & + & 1 \\
\vdots & & \vdots & & \vdots \\
n & = & n-1 & + & 1
\end{array}$$

We can say that the counting numbers are **generated by** 1. In other words, there is a *successor* function that takes a number and gives you the next one by adding 1.

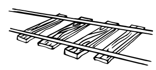
$$n \to \text{🎩} \to n + 1$$

2 The number line

A number line is a spatial representation of numbers. It allows us to map numbers to geometric insights and geometric insights to numbers. We will soon learn how to add and subtract quantities and how to easily determine if one number is greater than or less than another.

Fun Fact

You can think of the number line as a straight railway. It extends in both directions, but remains one-dimensional.



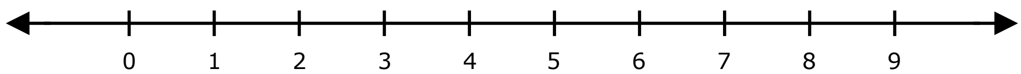


Figure 2.1: Number line with the numbers we have learned so far. The arrows indicate that the line extends infinitely in both directions.

2.1 Representing numbers on the number line

We can represent numbers on the number line as points, by marking the position of the number on the line. For instance, in the following figure, we have marked the positions of the numbers 3 and 7.

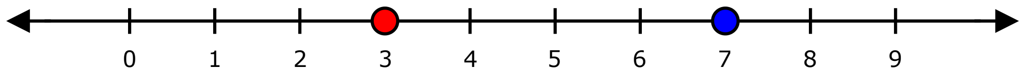


Figure 2.2: Representing numbers 3 and 7 on the number line.

We are going to use walking and traveling as a metaphor for moving on and adding in the number line, therefore we are taking some poetic liberties and are going to represent points and operations in a more playful way. For instance, the following figure represents number 2. You can think of it as the train is a position 2.

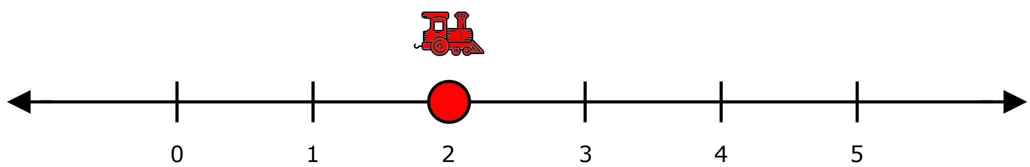


Figure 2.3: Representing number 2 on the number line.

2.2 Addition and subtraction on the number line

Adding up and subtracting quantities on the number line is very intuitive. Adding up means moving in the direction of increasing numbers, and subtracting means moving in the direction of decreasing numbers. I avoid using the words "right" and "left" or "east" and "west" because the line can be oriented or rotated in different ways.

For instance, imagine we want to do the following operation: $2 + 4$. We start 2:

Figure 2.4: We start by locating number 2 on the number line.

Then we move from there 4 units in the direction of the increasing numbers.

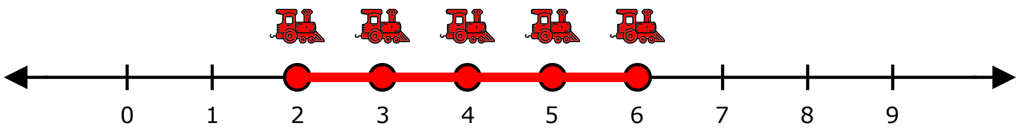


Figure 2.5: Starting at 2, we move 4 units in the direction of increasing numbers.

The position we end up at is the result, in this case $2 + 4 = 6$.

We will be representing the whole operation like this:

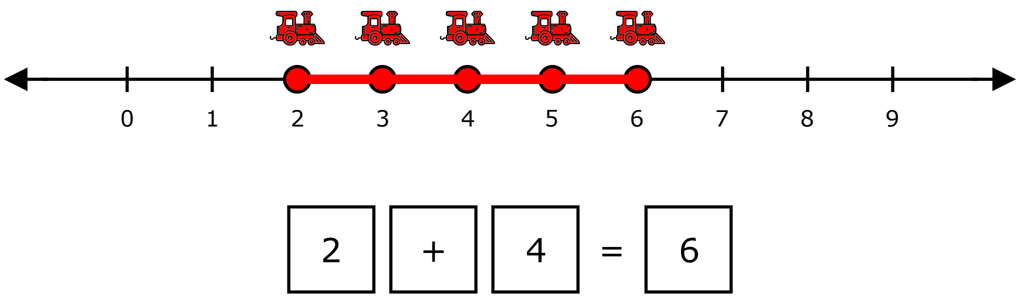


Figure 2.6: Computing the result of $2 + 4$ on the number line.

Subtracting is similar, only that we move in the direction of decreasing numbers. For instance, to compute $9-3$, we start at 9 and move 3 positions in the direction of decreasing numbers, and we end up at 6.

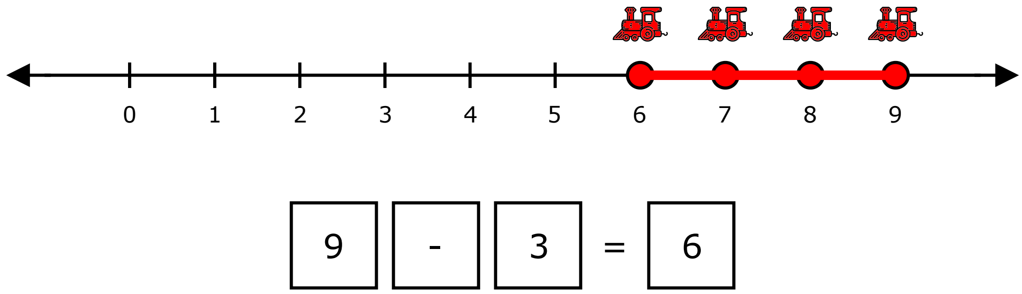


Figure 2.7: Computing the result of $9-3$ on the number line.

Time to practice. In the following exercises, use the number line to compute the result of the following operations. You have to start by coloring the train at the position of the first operand, and then move in the direction indicated by the operation, and finally write down the result in the box.

For instance, to compute $3+3$, we start at 3 and move 3 positions in the direction of increasing numbers, and we end up at 6.

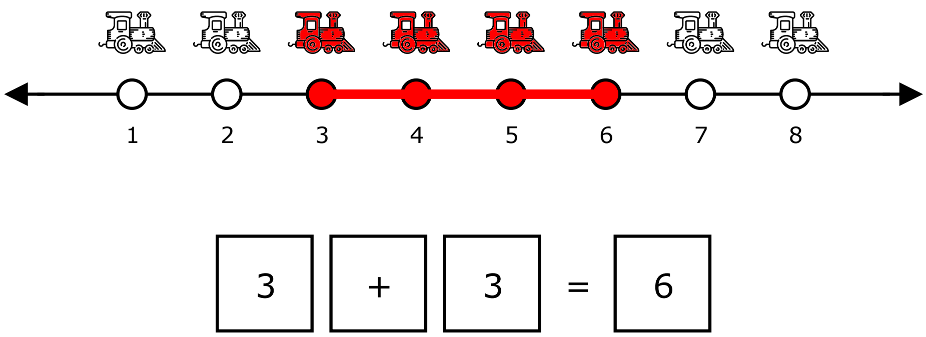


Figure 2.8: Compute $3+3$ on the number line.

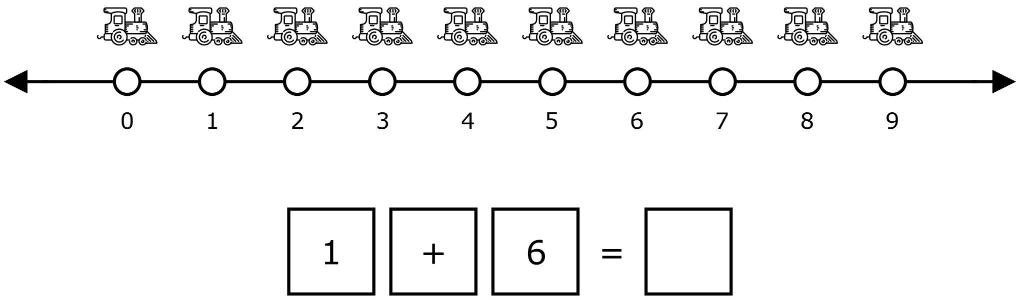


Figure 2.9: Compute $1 + 6$ on the number line.

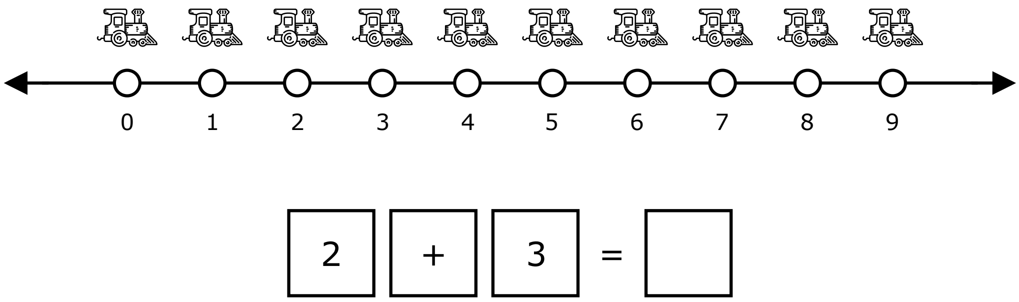


Figure 2.10: Compute $2 + 3$ on the number line.

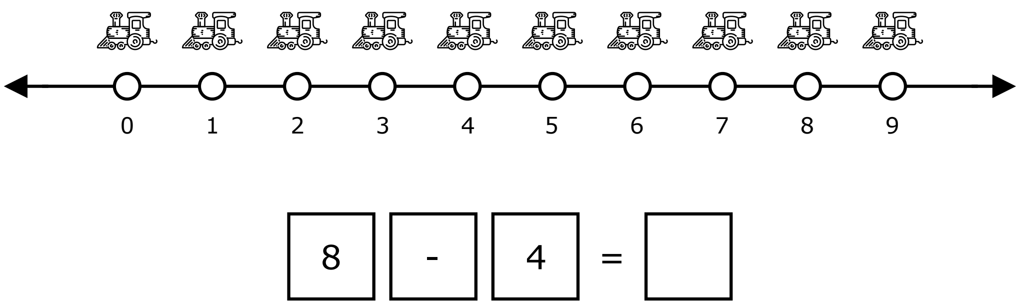


Figure 2.11: Compute $8 - 4$ on the number line.

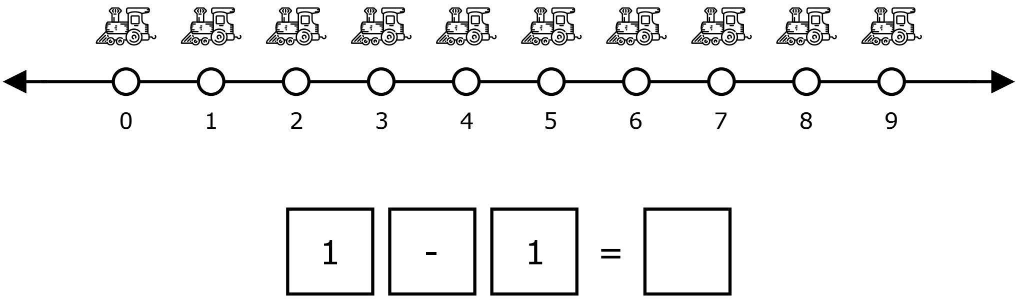
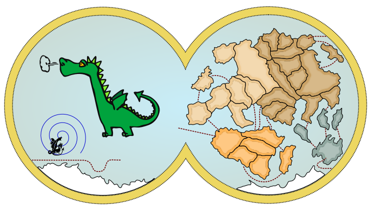


Figure 2.12: Compute $1 - 1$ on the number line.

Terra incognita

Old maps often labeled unexplored territories as "terra incognita," which is Latin for "unknown land." To signal the dangers of these areas, they were often illustrated with sea monsters and dragons. *Hic sunt dracones*: Here be dragons.



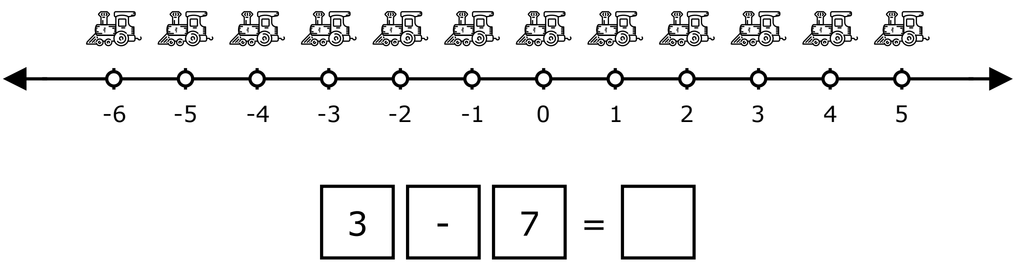


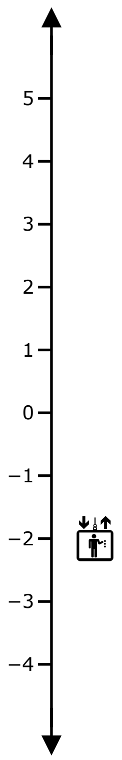
Figure 2.13: Compute $3 - 7$ on the number line.

Did you notice anything unusual about the last operation? We started at 3 and had to move 7 positions in the direction of decreasing numbers. However, after 3 positions, we ended up at 0. Does that mean we can't move any further? What is beyond 0 in that direction? *Suntne hic dracones?*  Not really. There are numbers in that direction that are pretty much like the ones you were familiar with on the positive side. These numbers are called negative numbers, and you can think of them as reflections of the positive numbers on the other side of zero. Each negative number is the reflection or *opposite* of the corresponding positive number. In the example above, after 0, you find the reflection of 1, which is -1 and is read as "negative one" or "minus one"; then, the reflection of 2, which is -2; and so on.

Fun Fact

You have likely encountered negative numbers when discussing cold temperatures (if you live in a place where the temperature can drop below 0 degrees) or riding an elevator. In elevators, positive numbers usually represent floors above ground level. Floors below ground level are usually labeled with negative numbers. This is convenient because it eliminates the need for a different display for floors below zero, like the basement.

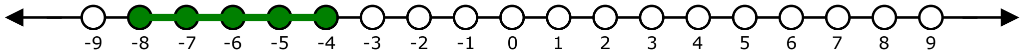
Negative numbers provide a convenient bookkeeping mechanism. You can use the same placeholder to represent quantities on two sides of a reference. They can represent deficits, losses, debts, or positions relative to an arbitrary reference. For example, you could represent a year prior to an event, an altitude below sea level, or a floor below the ground floor.



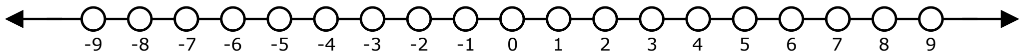
Exercises

Parent/educator Tip

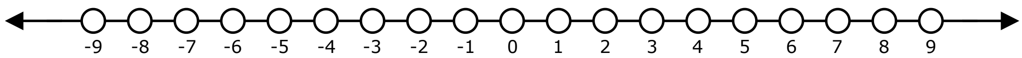
Use the following number lines as templates for additional exercises. Even better, have the kids create their own exercises by choosing the numbers and operations they want to practice.



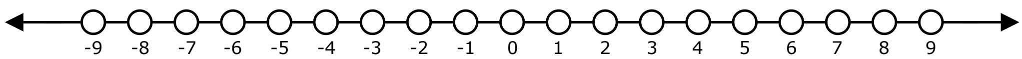
$\boxed{-8}\,\boxed{+}\,\boxed{4} = \boxed{-4}$



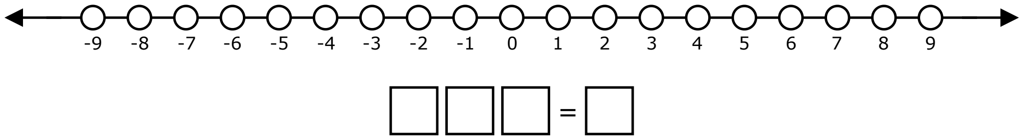
$\boxed{}\,\boxed{}\,\boxed{} = \boxed{}$

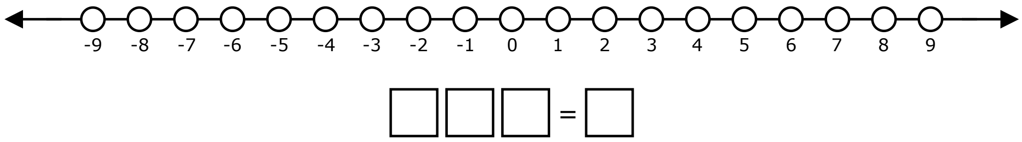


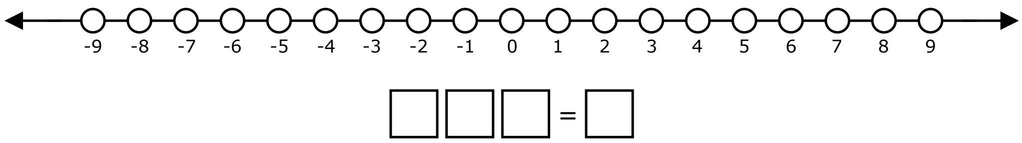
$\boxed{}\,\boxed{}\,\boxed{} = \boxed{}$

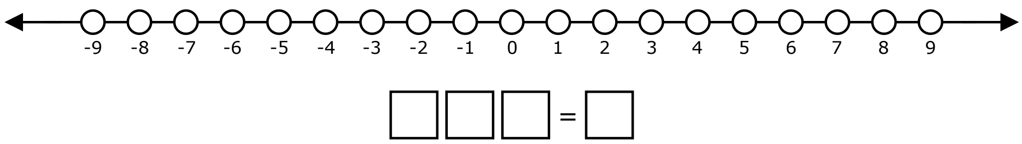


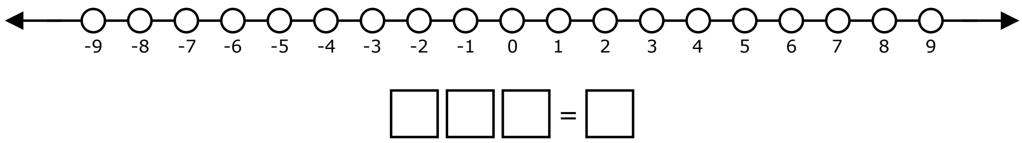
$\boxed{}\,\boxed{}\,\boxed{} = \boxed{}$


-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
=


-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
=


-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
=


-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
=



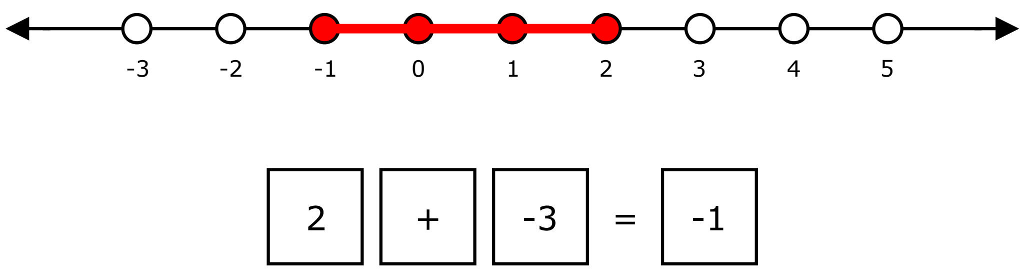
2.3 Negative numbers

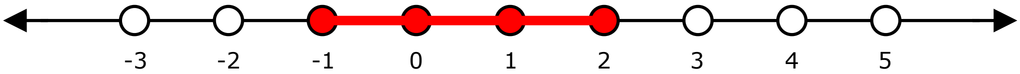
We learned that negative numbers are reflections of positive numbers on the opposite side of zero. They are similar to positive numbers, but represent different quantities. In the examples above, there is not a clear difference between positive and negative numbers on the number line other than their position relative to zero.

An important difference is that negative numbers, unlike their positive counterparts, are not used to represent the amount of an object. You don't usually say, "I have minus two apples." Perhaps because of this, negative numbers have been called "absurd" or "false" throughout history and were not widely accepted in the West until the Renaissance.

I just wanted to mention one last thing about how to perform operations with negative numbers.

Adding a negative number is the same as subtracting the corresponding positive number. For example, $2 + (-3) = 2 - 3 = -1$. This is more clearly seen on a number line. In both cases, the resulting operation involves moving 3 positions in the direction of decreasing numbers, starting from 2 and ending at -1.



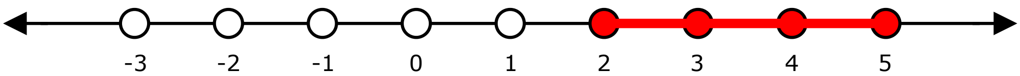


$$\boxed{2}\ \boxed{-}\ \boxed{3} = \boxed{-1}$$

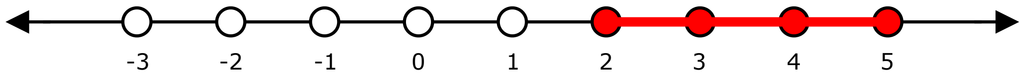
In the general case we have:

$$a + (-b) = a - b$$

Subtracting a negative number is the same as adding the corresponding positive number. For example, $2-(-3) = 2+3 = 5$. In both cases, the resulting operation involves moving 3 positions in the direction of increasing numbers, starting from 2 and ending at 5.



$$\boxed{2}\ \boxed{-}\ \boxed{-3} = \boxed{5}$$



$$\boxed{2}\ \boxed{+}\ \boxed{3} = \boxed{5}$$

In the general case we have:

$$a - (-b) = a + b$$

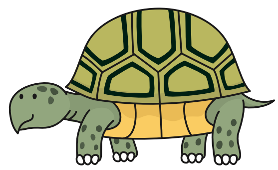
This last result may be less intuitive. One way to understand it is that each minus sign changes the direction. For example, when you have $a-(-b)$, the first minus sign, $-$, tells us to change direction and move in the direction of the decreasing numbers. However, the second minus sign, $-$, tells us to change direction again and move in the direction of the increasing numbers. That's why we end up moving b positions in the direction of increasing numbers.

Still confused? No worries—we'll revisit this topic later.

2.4 How far apart are the tortoise 🐢 and the hare 🐇?

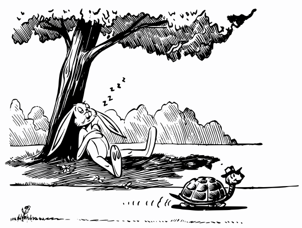
Turtles all the way down

Tortoises and turtles are recurrent characters in philosophy and mathematics. In many traditions, a giant world turtle supports the earth. One of Zeno's paradoxes is called "The Tortoise and Achilles" and intended to illustrate the impossibility of motion. There is also Aesop's fable "The Hare and the Tortoise." Though not mathematical in nature, we will use the characters from this fable to illustrate the concept of distance.



The hare and the tortoise

A Hare was one day making fun of a Tortoise for being so slow upon his feet. "Wait a bit," said the Tortoise; "I'll run a race with you, and I'll wager that I win." "Oh, well," replied the Hare, who was much amused at the idea, "let's try and see"; and it was soon agreed that the fox should set a course for them, and be the judge. When the time came both started off together, but the Hare was soon so far ahead that he thought he might as well have a rest: so down he lay and fell fast asleep. Meanwhile the Tortoise kept plodding on, and in time reached the goal. At last the Hare woke up with a start, and dashed on at his fastest, but only to find that the Tortoise had already won the race.



Read more Aesop's fables at Project Gutemberg: https://www.gutenberg.org/

Measuring distance on a number line is simple. It boils down to counting the number of units or segments of lenght 1 between two numbers. These units are the space between two consecutive numbers; for example, between -2 and -1, or between 3 and 4.

What is a distance?

A distance is a numerical measurement of how far apart two points or objects are. Examples include the distance from your house to school, the distance from Earth to the sun, the number of days until Christmas, and how many moves your Rubik's Cube is from being solved.

Distances are nonnegative numbers and are symmetric, meaning the distance between your house and the playground is the same as the distance between the playground and your house.

distance (🏡, 🛝) = distance (🛝, 🏡)

distance (🌎, 🌞) = distance (🌞, 🌎)

The distance between an object and itself is always zero. For instance, the distance between your house and your house is zero, and the distance between the sun and the sun is zero.

distance (🏡, 🏡) = 0

For example, in the following figure, we want to find the distance between the hare 🐇 and the tortoise 🐢. This is how it is done:

- First, we identify the position of the hare, which is 1.
- We identify the position of the tortoise, which is 5.
- We count the number of segments between the two numbers (colored fuchsia in the figure), which is 4.
- An equivalent way to find the distance is to subtract one number from the other, i.e., $5 - 1 = 4$.

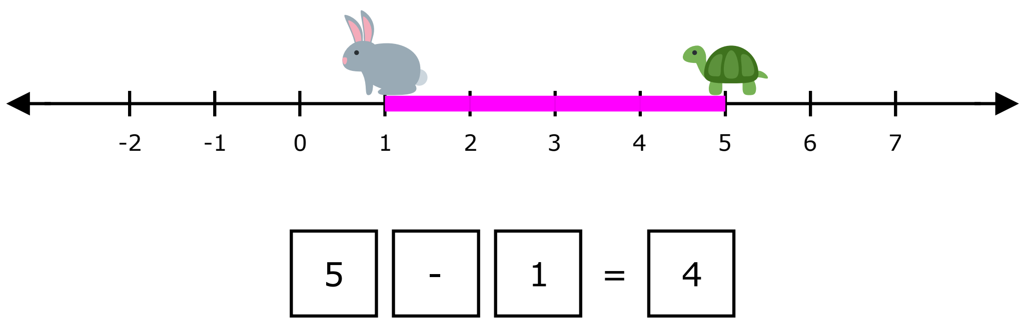


Figure 2.14: Measuring distance on the number line.

If the hare moves to position -3, that's fine; the same logic applies. You just need to remember how to subtract negative numbers. Remember that $5-(-3) = 5+3 = 8$

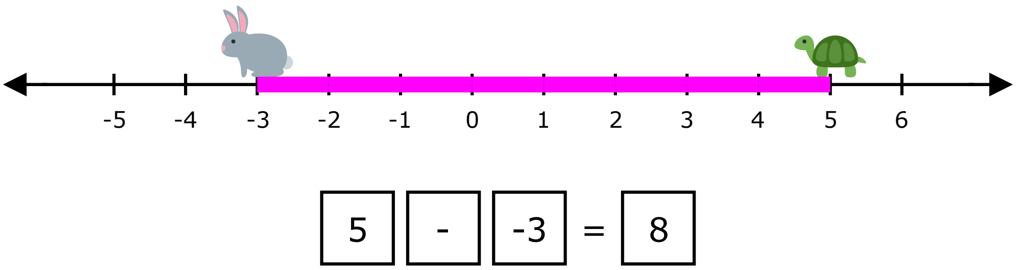
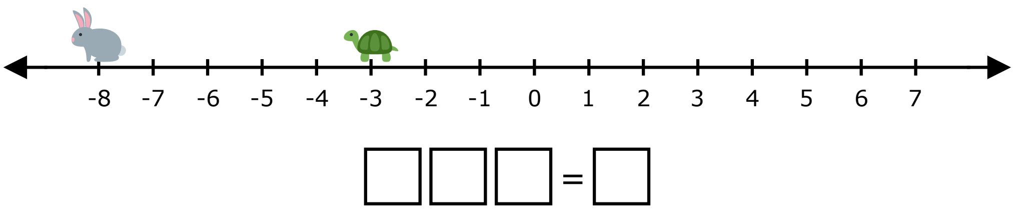
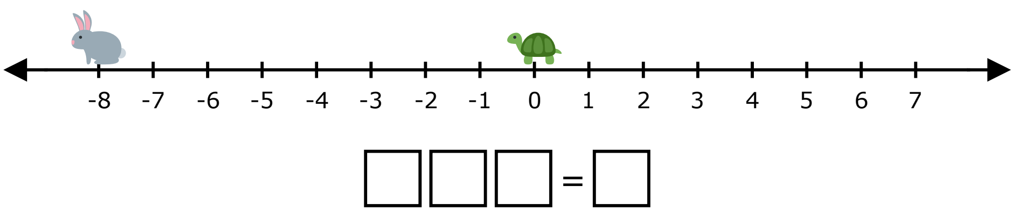


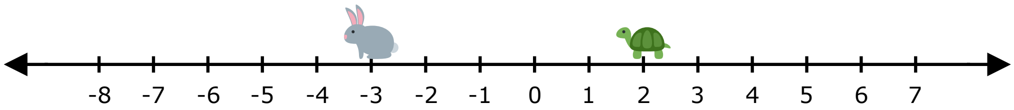
Figure 2.15: Measuring distance on the number line.

Exercises

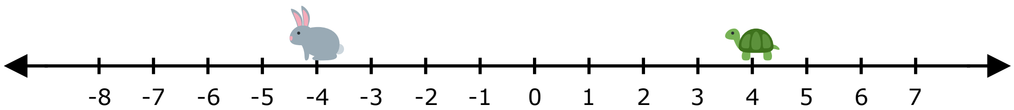
Find the distance between the hare and the tortoise:



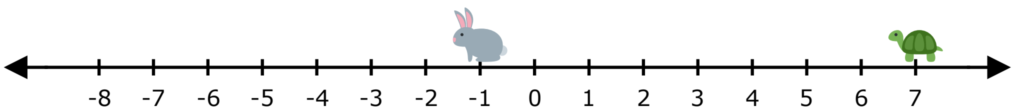




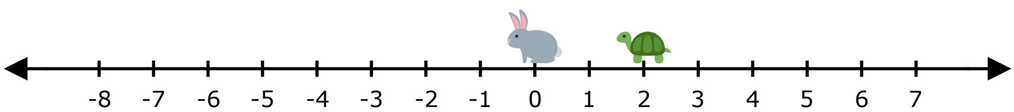
$\square\square\square=\square$



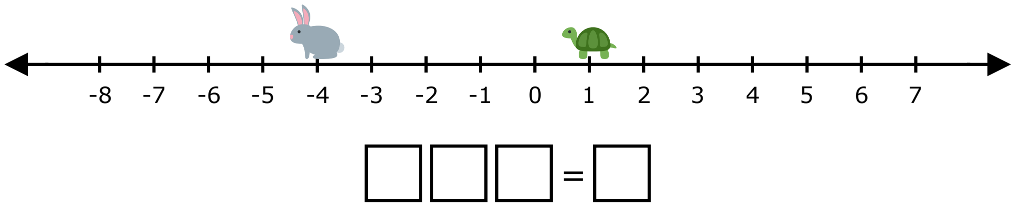
$\square\square\square=\square$



$\square\square\square=\square$



$\square\square\square=\square$



2.5 Comparing quantities on the number line

Once we have established the direction in which the numbers increase, comparing quantities on the number line is trivial.

Thus far, we have drawn horizontal number lines and agreed that numbers increase as we move towards the right.

Therefore, given two numbers, the one on the right is always larger than the one on the left.

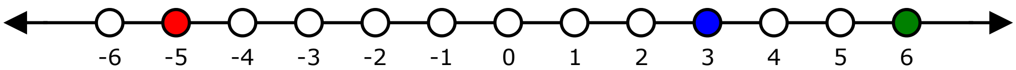


Figure 2.16: Comparing quantities on the number line. $-5 < 3 < 6$.

2.6 Playing Battleship 🏴‍☠️ 🦜

Flatland

Mathematical objects can exist in any number of dimensions. You can think of a dimension as one of the directions in which you can move. Because we live in a three-dimensional world, we can move left or right, forward and backward, and up and down.
Let's imagine how it would be to live in a world with different number of dimensions N.

N	Object	Symbol	How can we move?
0	Point	•	We cannot move.
1	Line		We can move only forward and backward.
2	Plane		We can move forward and backward, and also left and right.
3	Volume		We can move left, right, forward, backward, and jump up and down.

More than 3 dimensions are hard to imagine and visualize.

So far, we have been operating and counting in one dimension. This means that we can only move in one of the two directions in which the arrows of the number line extends: for instance right and left.

We can use two perpendicular number lines to create a two-dimensional plane. Perpendicular lines look like this: $\perp$.

The resulting 2-dimensional plane is called the **Cartesian plane**. It is named after the French philosopher and mathematician René Descartes, whose latinized name was Renatus Cartesius.

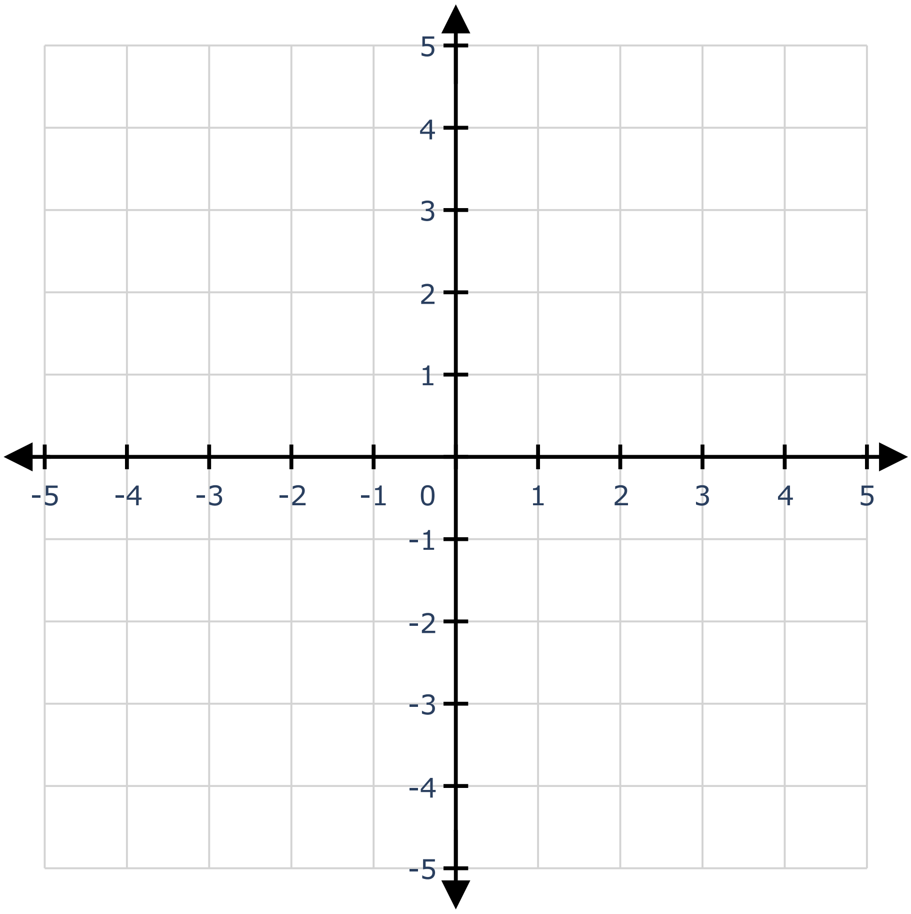
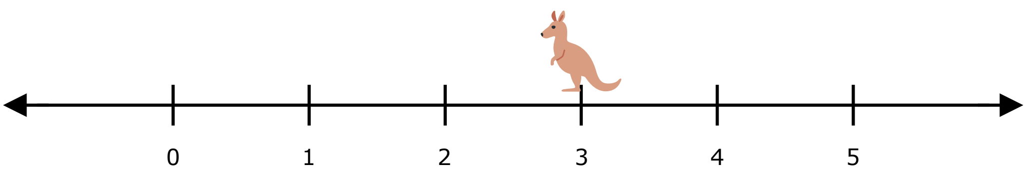


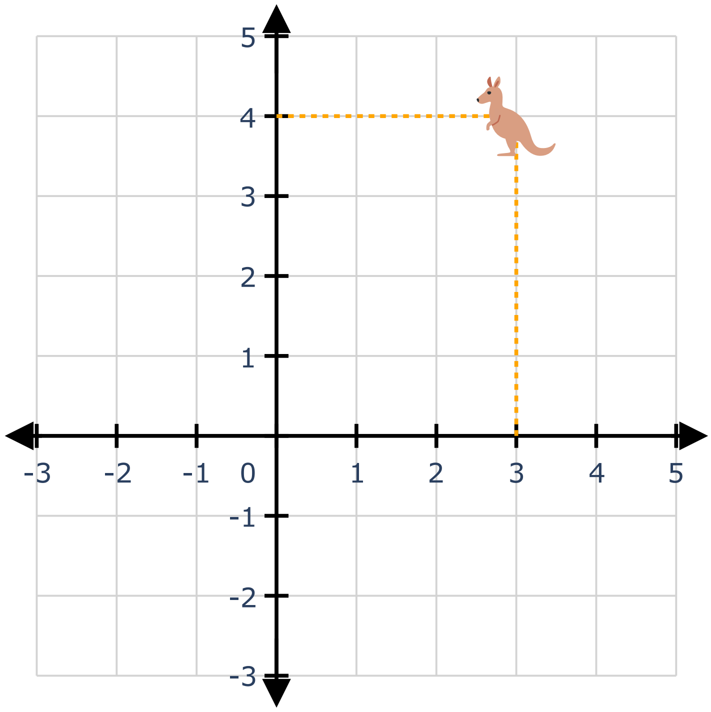
Figure 2.17: Cartesian plane built with two perpendicular number lines.

In the number line, we could move only in two directions, and specifying the position of a point was easy: we needed just one number.



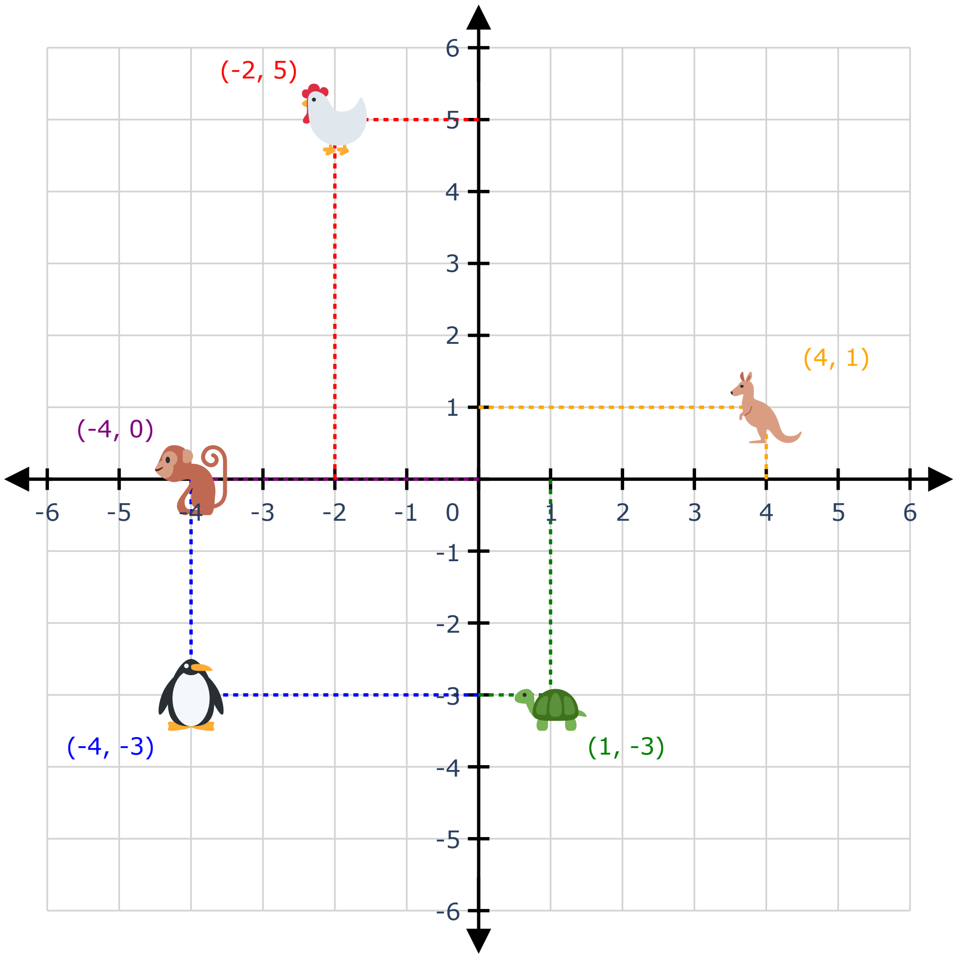
For instance, we could say the kangaroo is at position 3 on the number line.

Now, in two dimensions, we can move not only left and right, but also in the perpendicular direction. Let's call it for convenience "up and down".



Now to specify the position of the kangaroo, we need two numbers: one for the horizontal direction and one for the vertical direction. We can say that the kangaroo is at position $(3, 4)$, where 3 is the horizontal coordinate and 4 is the vertical coordinate. The order is important.

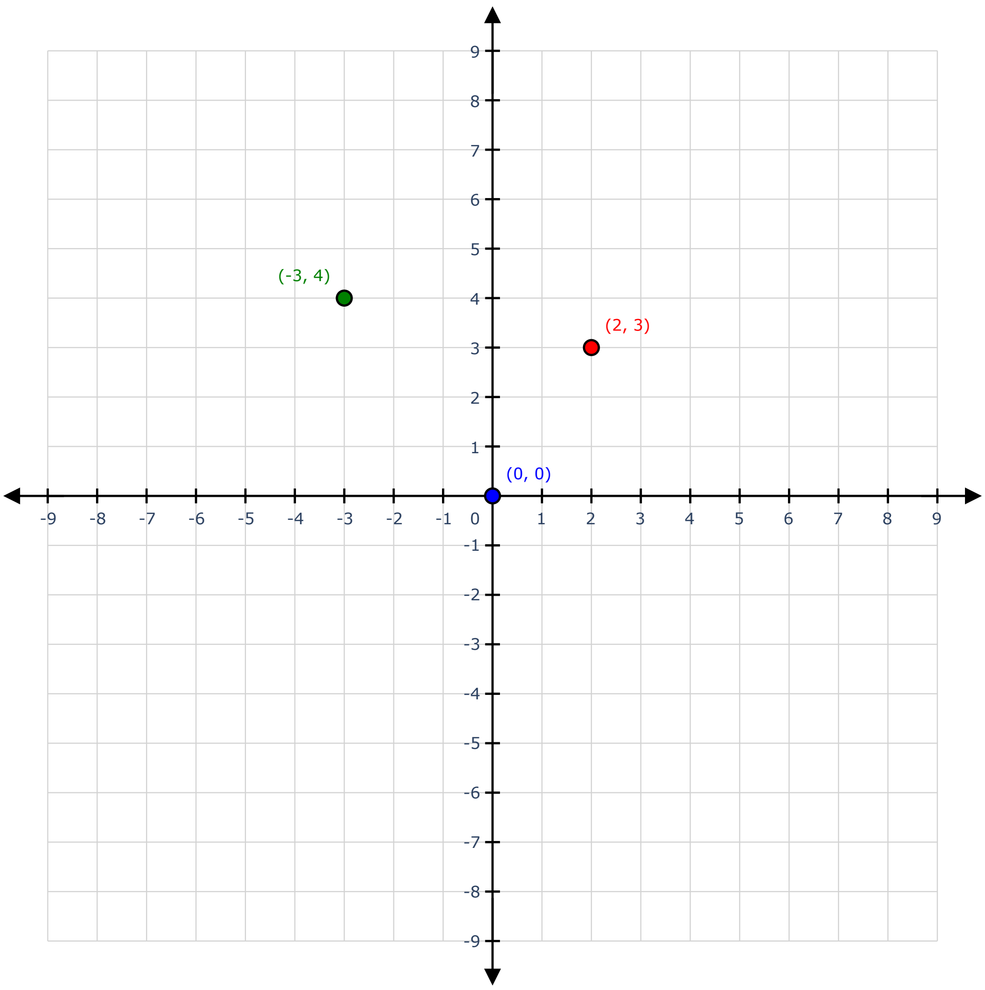
Coordinates can also be negative. In the following figure, the coordinates are shown close to each animal. For instance, the penguin is at $(-4, 3)$, the monkey is at $(-4, 0)$, etc.



(-2, 5)
(4, 1)
(-4, 0)
(-4, -3)
(1, -3)
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6

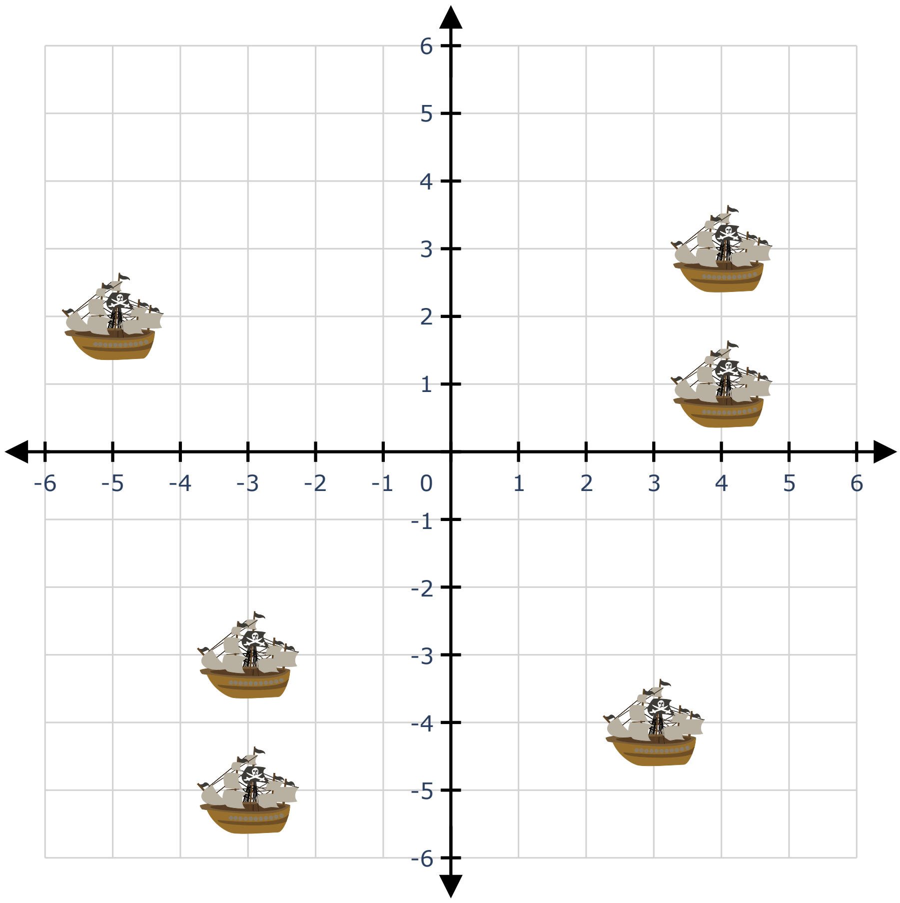
Exercises

Locate the following points one the Cartesian plane: $(2,3)$, $(0,0)$, $(-3,4)$, $(-2,5)$, $(1,-3)$, $(0,-7)$, $(-9,0)$, $(-9,-9)$, $(-9,1)$, $(0,7)$, and $(1,1)$.

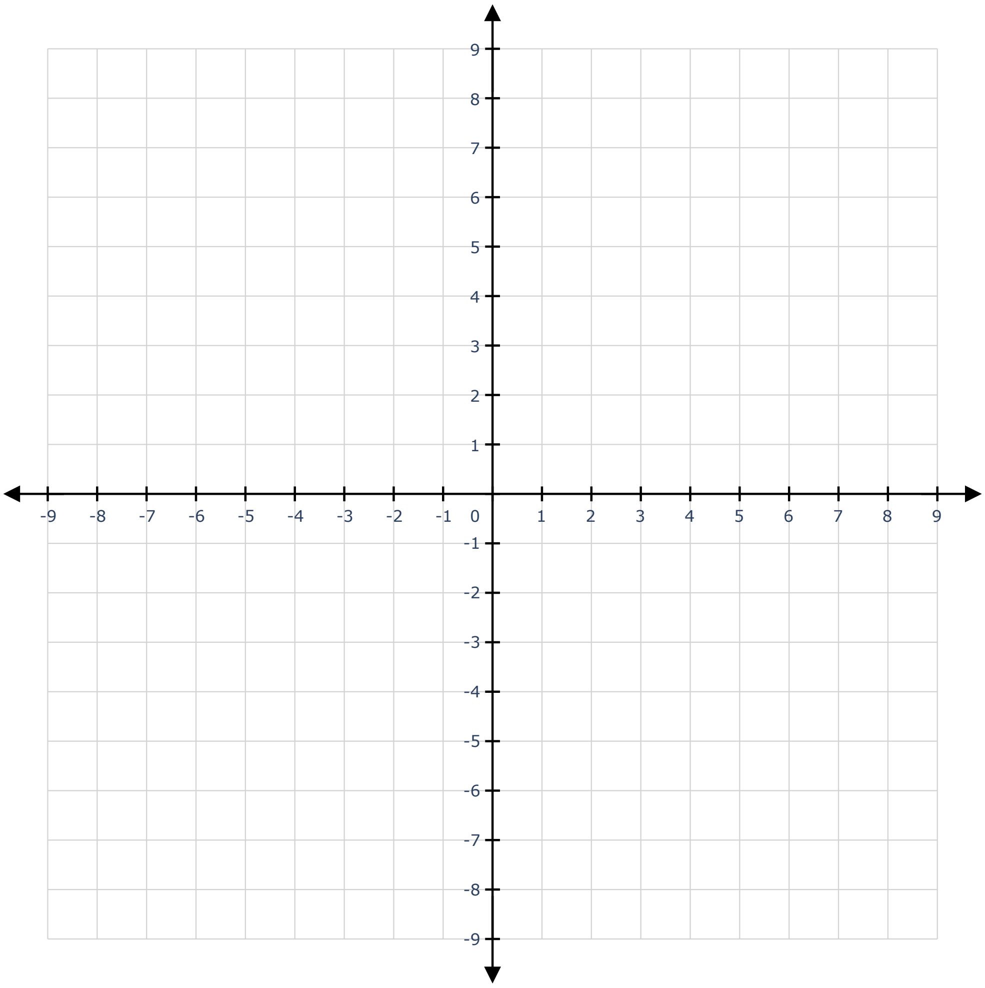


Battleship

You can play Battleship on a Cartesian plane. Each player must secretly place their ships on the grid. Then, players take turns calling out coordinates in an attempt to hit their opponent's ships. The first player to sink all of their opponent's ships wins.



Print two copies of the figure below and play Battleship with a friend.



2.7 The integers

The counting numbers we introduced in Chapter 1 and have used so far form a **set**: the set of natural numbers.

$$\mathbb{N} = \{0, 1, 2, 3, 4 \cdots\}$$

> **What is a set?**
>
> A set is a collection of objects. For instance, this a set of animals:
>
> $A = \{$ 🐨, 🐶, 🐸, 🐼, 🐔 $\}$
>
> A set can be a subset of another set, meaning that it is contained in that set. For instance, the set of mammals, which are hairy, warm-blooded vertebrates that drink milk, give birth to live young:
>
> $M = \{$ 🐨, 🐶, 🐼 $\}$
>
> is a subset of the previous set of animals, A. We denote it using the symbol $\subset$, that is $M \subset A$.
> Some sets contain only 1 element, like the set of moons of planet Earth $S = \{$ 🌔 $\}$. Some sets contain no elements, like the set of moons of planet Mercury or the set of sharks that can fly $S' = \{\}$. This set is called the empty set and is denoted with $\emptyset$.

The set that results from extending the natural numbers with the negative reflections of the positive numbers is called the **integers** and is denoted with $\mathbb{Z}$.[1]

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

This new set contains all the numbers we have learned so far, including the natural numbers and the negative numbers.

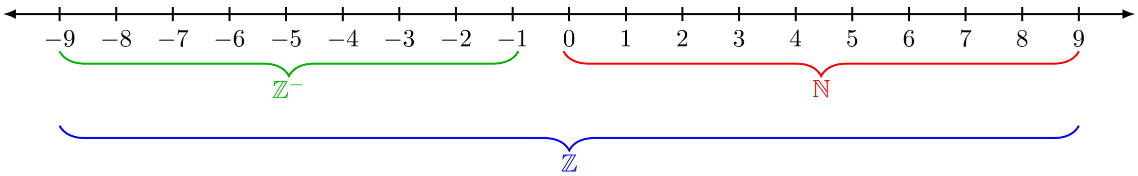
[1] $\mathbb{Z}$ stands for "Zahlen", the German word for "numbers".

$$\color{red}{\mathbb{N}} \subset \color{blue}{\mathbb{Z}}$$

The negative integers are also a set denoted with $\mathbb{Z}^-$.

$$\color{green}{\mathbb{Z}^-} \subset \color{blue}{\mathbb{Z}}$$

These relationships are shown in the following figure.



$\mathbb{N}$ and $\mathbb{Z}$ are very important sets in mathematics, you will find them over and over again in different contexts.

$\mathbb{Z}$ and the operation of addition on this set has some important properties:

- ✅ **Closure and associativity**: Combining two elements in the set yields another element in the set. If we add two integers, the result is another integer. For example, $2 + 3 = 5$ and $-1 + (-7) = -8$.
- ✅ **Existence of an identity element**: There is a *neutral* number e, such that $e + a = a + e = a$. In the case of addition, this do-nothing number is 0. For example, $2 + 0 = 0 + 2 = 2$ and $-3 + 0 = 0 + (-3) = -3$. Adding (or subtracting) 0 is like doing nothing.
- ✅ **Existence of an inverse**: For every integer a, there is another integer that, when added to the first, yields the identity element 0. For example, the additive inverse of 2 is -2 because $2 + (-2) = 0$, and the additive inverse of -3 is 3 because $-3 + 3 = 0$. You can think of the additive inverse as the evil twin of a number, the reflection of the number across zero.

These properties make the set of integers with the operation of addition a mathematical structure called a **group**.

Whoa, why are groups important?

It's important to establish that the integers form a group under addition because all future group concepts will apply to the integers under addition.

Here is a loose analogy: You know that mammals are warm-blooded vertebrates that drink milk, give birth to live young, and have hair.

Then I ask you about okapis and saolas. You might not know what they are, as I didn't until recently. But if I tell you that okapis and saolas are mammals, suddenly you know a lot about them.

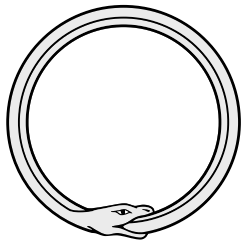
2.8 Number lines that eat their own tails

> Dad doesn't like people who talk in circles, but he loves people who count in circles.

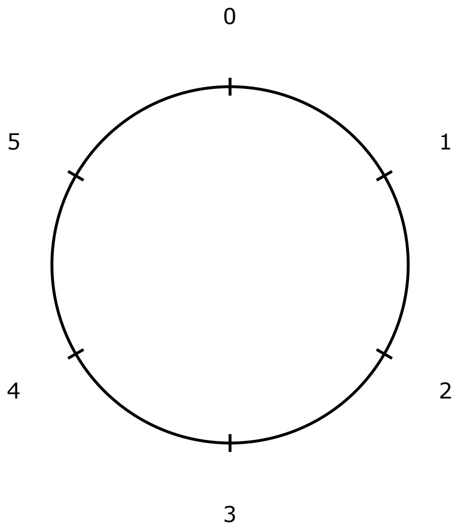
I told you before that we can orient the number line in any way we want. It turns out, we can also bend it in a circle.

Ouroboros

Have you seen those depictions of snakes or dragons eating their own tails? They are called "ouroboros" and they represent the idea of something that is cyclic. "οὐροβόρος" is ancient Greek for "tail eater". Now, let's see what we can do with a number line that eats its own tail.

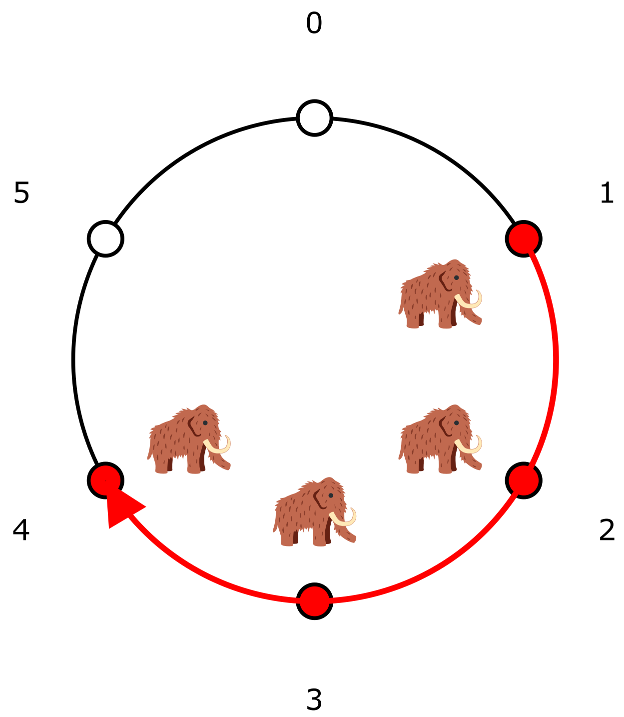


We can take a number line and bend it in a circle. Isn't that cool?



The kind of arithmetic we can do on a circular number line is called *modulo arithmetic* or *clock arithmetic*, because it is the kind of arithmetic we use when we read a clock.

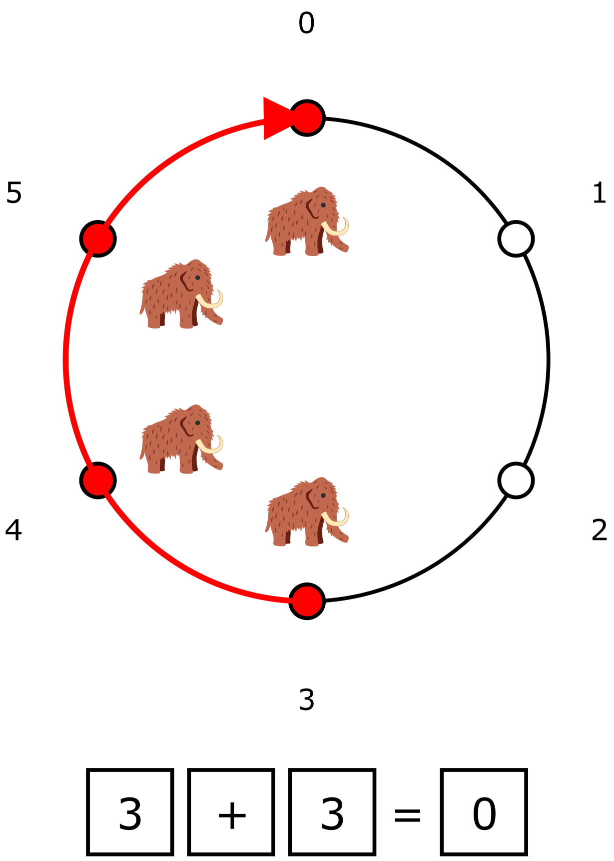
Let's consider how to make an addition in this circular number line. Imagine we want to calculate $1+3$. We start at 1 and move 3 steps in the clockwise direction.



$$\boxed{1}\ \boxed{+}\ \boxed{3} = \boxed{4}$$

That's pretty normal. Expected.

Now consider we want to do this addition $3 + 3$. We start at 3 and move 3 steps.



$3 + 3 = 0$. That's new ah? 🤯

This number system is called $\mathbb{Z}_6$. Here is the table of operations for addition in $\mathbb{Z}_6$:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

If you want to calculate a sum in $\mathbb{Z}_6$, for instance $4 + 3$, you can find the row for 4 and the column for 3 and see where they intersect.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Figure 2.18: $4 + 3 = 1$

Same as $\mathbb{Z}$, $\mathbb{Z}_6$ is also a group.

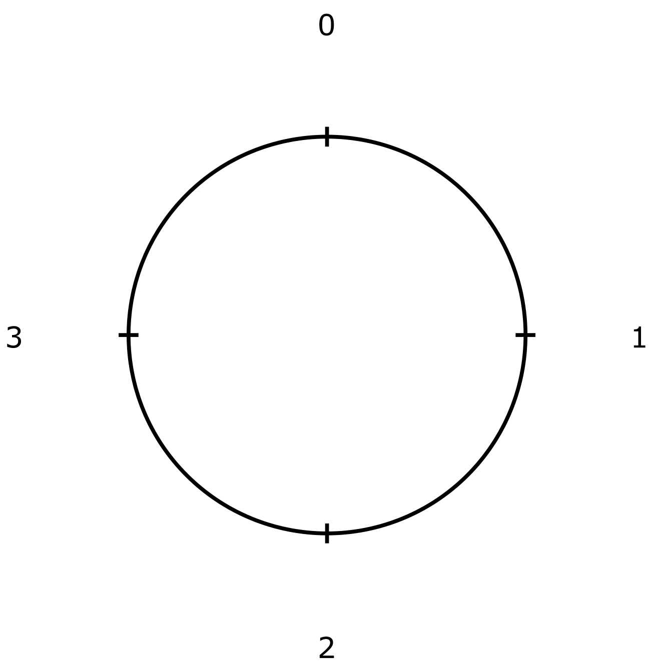
Why?

Same checklist we used before for $\mathbb{Z}$:

- ✅ **Closure and associativity**: If you add two integers in $\mathbb{Z}_6$, you get another integer in $\mathbb{Z}_6$. For example, $2+3=5$ and $4+3=1$. The operations are associative, for example, $(2+3)+4=2+(3+4)=5+4=2+1=3$.
- ✅ **Existence of an identity element**: There is a number, 0, such that adding it to any number doesn't change that number. For example, $2+0=2$.
- ✅ **Existence of an inverse**: For every integer, there is another integer such that when you add them together, you get the identity element 0. For example, for 2, the inverse is 4 because $2+4=0$, and for 3 it is 3 because $3+3=0$.

Exercises

Now consider the group $\mathbb{Z}_4$ with elements $\{0,1,2,3\}$ and addition modulo 4.



Complete the group table for addition in $\mathbb{Z}_4$:

+	0	1	2	3
0	0	1	2	3
1				
2				
3				

3 Counting beyond 9

There is nothing to stop us counting beyond 9. Chances are you already know how to do it and won't have any trouble answering the following questions. If you still cannot, just skip and come back to them after reading the rest of the chapter.

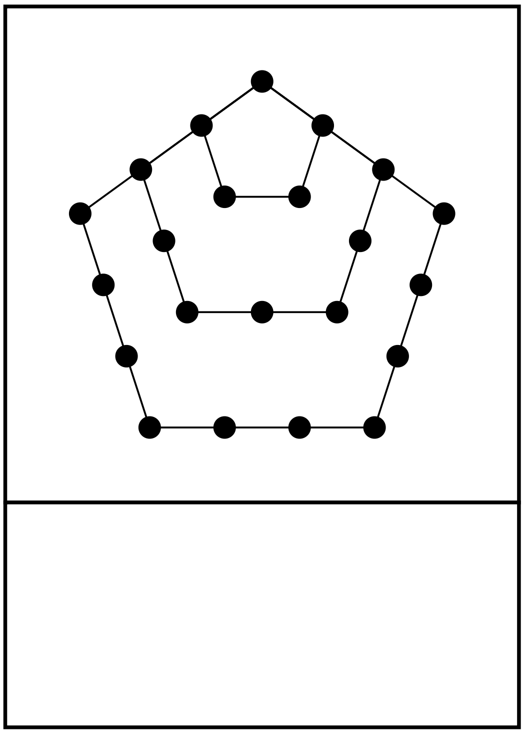


Figure 3.1: Count the number of points in this pentagonal number.

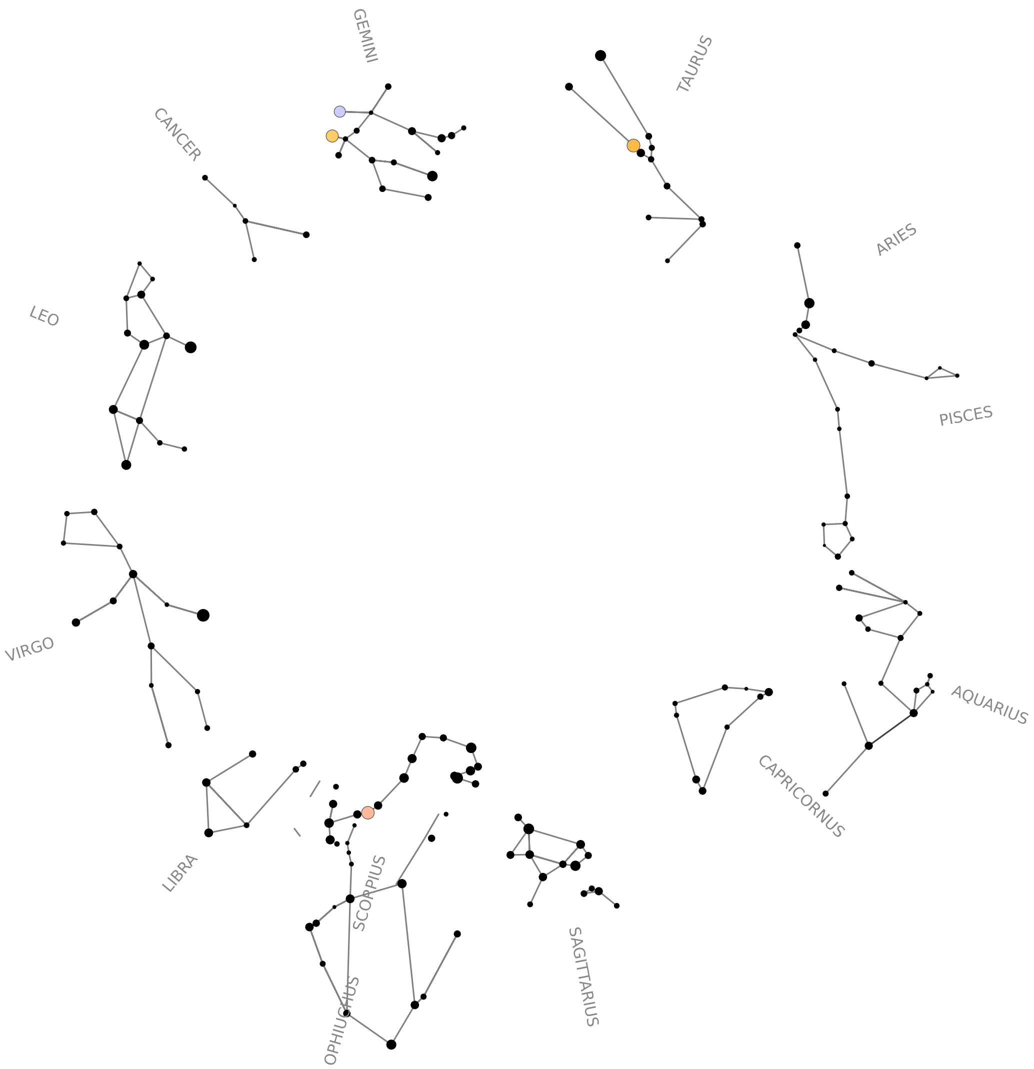


Figure 3.2: Count the number of constellations in the Zodiac.

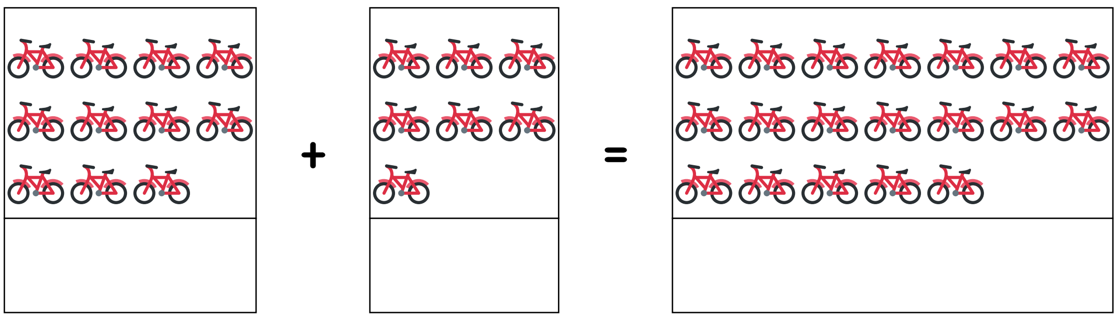


Figure 3.3: Find the total number of bikes.

However, the problem may lie in the way numbers are represented. In this chapter, we will learn how to represent and construct numbers larger than 9 using the decimal system.

3.1 How could we represent numbers greater than 9?

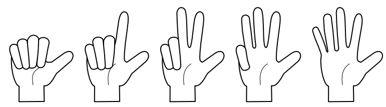
If we only have ten *digits* or symbols:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Figure 3.4: The decimal numbers. Ten digits.

Decem digiti

The word "digit" comes from the Latin word "digitus," meaning "finger." The plural form is "digiti."



Question

How to possibly represent numbers above 9? Any ideas?

If we don't want to introduce new symbols, it seems the only option is to reuse the same symbols in some way 🤔. We need to exploit a different property or attribute of these digits somehow. Any ideas?

What about their color? We could establish a color convention where the value of the digitgs depends on their color.

9, 9, 9, 9

Instead of color, we could use another attribute, such as size.

9, 9, 9, 9

If you quickly scan the page numbers in this book, you will see that this is not how it is done. Instead, we use the **position** of the digits.

Just as we can use the letters of the alphabet to write an infinite number of sentences and books, we can also use the ten decimal digits to represent infinite many numbers.

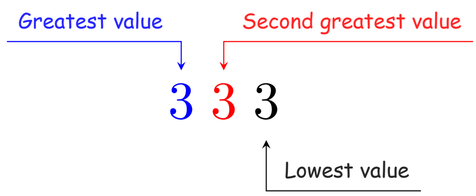
3.2 Positional number system

Numbers greater than 9 can be represented by combinations of the same ten digits in different positions. Any combination that starts with a number other than 0 is a valid number. Let's see some examples:

$$10, \quad 123, \quad 9999999, \quad 100000000000, \quad \ldots$$

Positional number system in a nutshell

The defining feature of the positional number system is that the value of a digit depends on its position in the number. For instance, in the number 333, each digit 3 has a different value.



The farther to the left ⇐ a digit is, the larger its value. We will use boxes to emphasize the positions.

3	3	3

A digit's value changes as soon as it is moved to a different position. For example, the resulting number changes each time the digit 3 is moved to the left; the larger the position to the left, the larger the number.

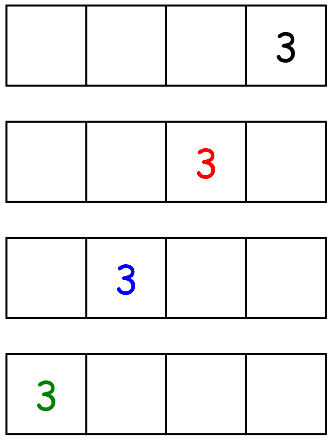
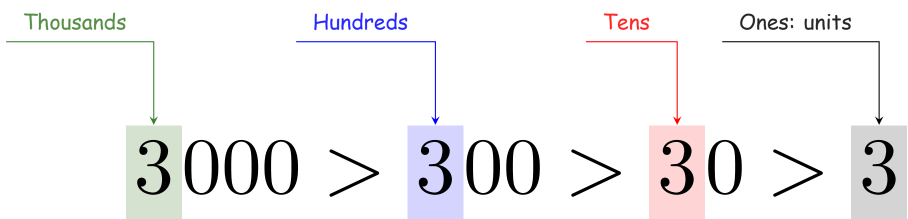


Figure 3.5: A digit's value depends on its position. It increases as we move to the left ←.

However, in the last example, these are not valid numbers. We need to fill the empty positions to the right.

Figure 3.6: The digit 3 has different values depending on its position.

You don't need to know what the numbers are; you only need to know that the further to the left the digit 3 is, the larger the value it represents[1]:



Even if you don't know what the numbers mean, you can use what you just learned to determine which number is larger and who is faster in the figure below.

Explain why?

[1] We don't need to draw the boxes. What matters is the position of each digit.

How to count and construct numbers beyond 9?

We now how to count until 9:

0 - 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9

To count beyond 9 we need two things: more positions and reusing the same digits.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19

Notice that after 9 we start reusing the same digits, beginning with 0, but now they are prefixed by the digit 1.

> **Question**
>
> Can you guess what number comes after 19?

00	01	02	03	04	05	06	07	08	09
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49

As you probably guessed, after 19 comes 20. Each time we reach a number ending in 9, we increase the leftmost digit by 1. If you only look at the red digits, you will see that we are also counting in that position: $0, 1, 2, 3, 4, \cdots$. Here, we prefixed the first ten numbers with 0 to emphasize that, but that doesn't change the values.

> **Question**
>
> How high do you think we can count with two positions?

If you answered 99, you are correct!

80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

> **And now what?**
>
> How to continue after 99?

The answer is always the same: we need two things —more positions and reusing the same digits.

080	081	082	083	084	085	086	087	088	089
090	091	092	093	094	095	096	097	098	099
100	101	102	103	104	105	106	107	108	109
110	111	112	113	114	115	116	117	118	119

Notice that with the blue digits, we are also counting in the third position: $0, 1, 2, 3, 4, \cdots$. Only, we count slower than we did with the red digits. The most to the left we go, the slower we count.

This is a recurring process. We continue to do the same thing: add positions and reuse the same digits to create larger and larger numbers.

Exercises

- Write the numbers from 0 to 31.

0, 1, 2, 3

- Write the numbers from 90 to 110.

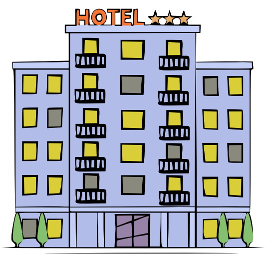
90, 91, 92, 93, 94

In order to master these insights and the way numbers are constructed, we are going on vacation! We'll be staying at the Numbers Grand Hotel.

3.3 The Numbers Grand Hotel

Fun Fact

I am probably the first person to use the analogy of a Grand Hotel to explain mathematical concepts. Our Grand Hotel is a special one with ten rooms on each floor. The owner is a reasonable man, which is why the floor and room numbers start at zero.



Our hotel rooms look like this. They have a door and a number on top of it.

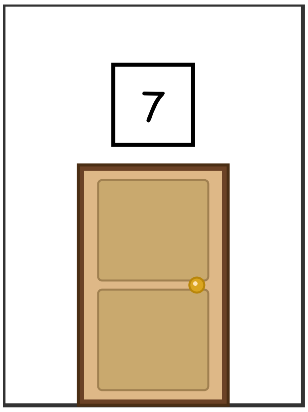


Figure 3.7: Room number 7.

The numbers on top of the door are very useful to identify the room. If our hotel has only one floor, we can easily identify a particular room with a single digit number.

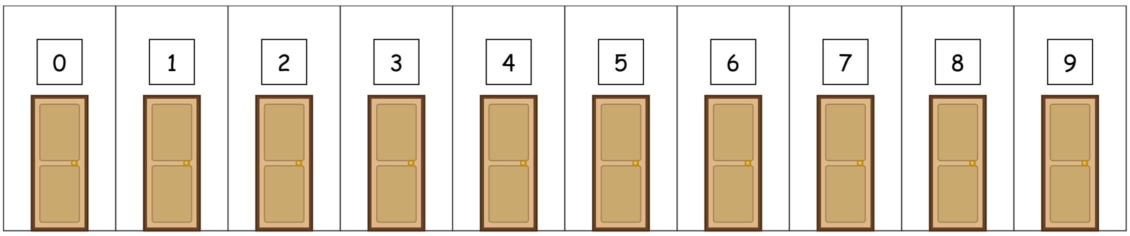


Figure 3.8: Ground floor (floor 0) of our Grand Hotel.

You wouldn't have any problem finding room number 7 in the first floor, right? But what happen if our hotel has 2 or more floors?

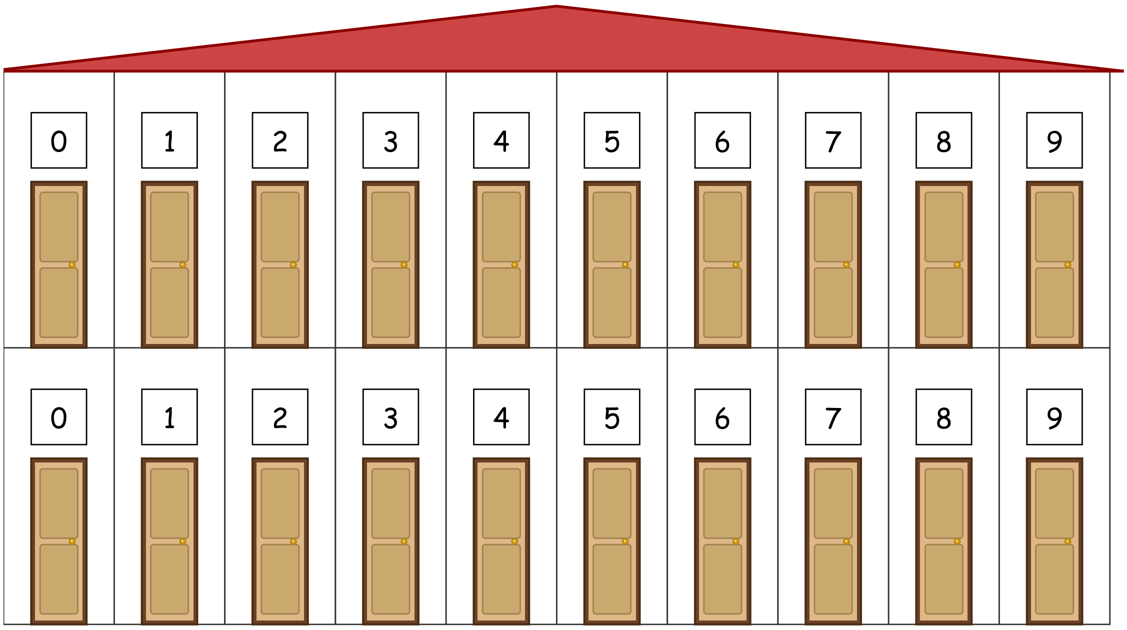


Figure 3.9: The first two floors of our Grand Hotel. Rooms have single-digit numbers.

You cannot unambiguously identify Room 7 using single-digit numbers. Clearly, we need more numbers to represent additional rooms, and you already know how to construct larger numbers. As we saw in the previous section, the first thing we need to do to represent larger numbers is to add more positions.

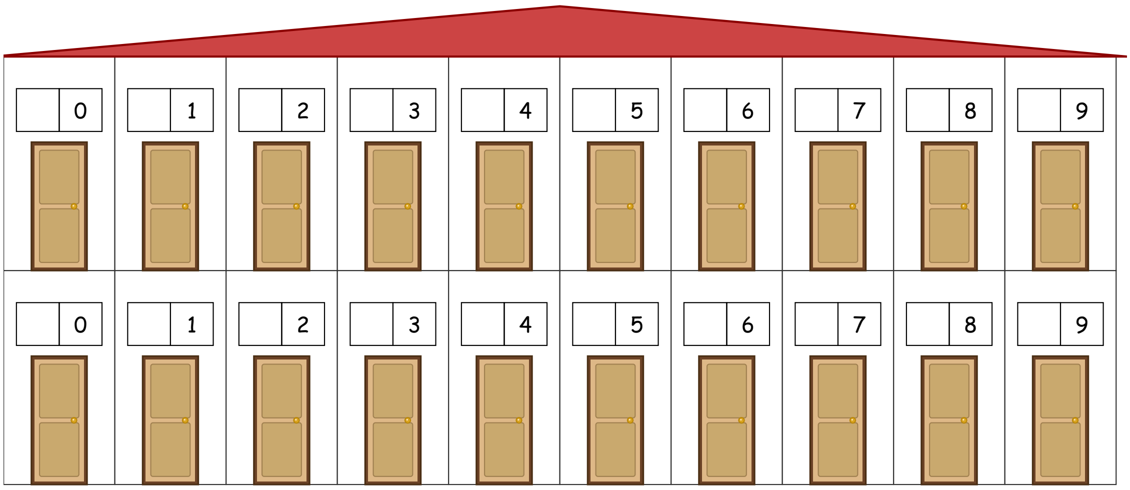


Figure 3.10: The first two floors of our Grand Hotel. Each room number has now two placeholders (two boxes).

Question

Adding one position is not enough; we also need to fill them with digits. Which digits should we use to unambiguously identify a particular room?

That's right; you also need to specify the floor number. For example, you could say "Room 7 on the 0 floor" or "Room 7 on the 1 floor."

We will now add a second digit to each door in the second placeholder to represent the floor number. The first position, the rightmost one, is for the room number; the second position is for the floor number.

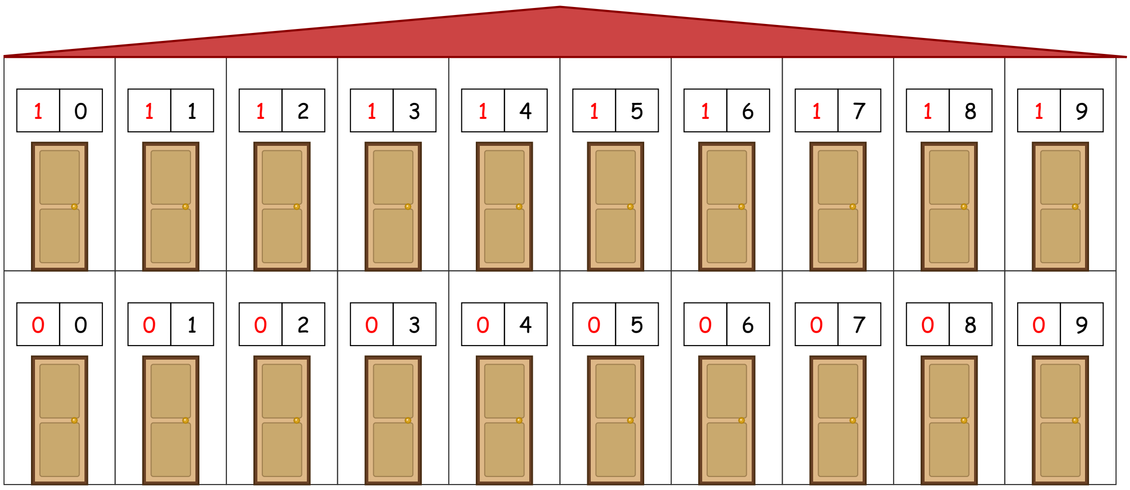


Figure 3.11: First 2 floors (floor 0 and floor 1) of our Grand Hotel.

In the first floor, we don't really need those padding zeros in front, so we can safely remove them. The following numbering is equivalent to the previous one.

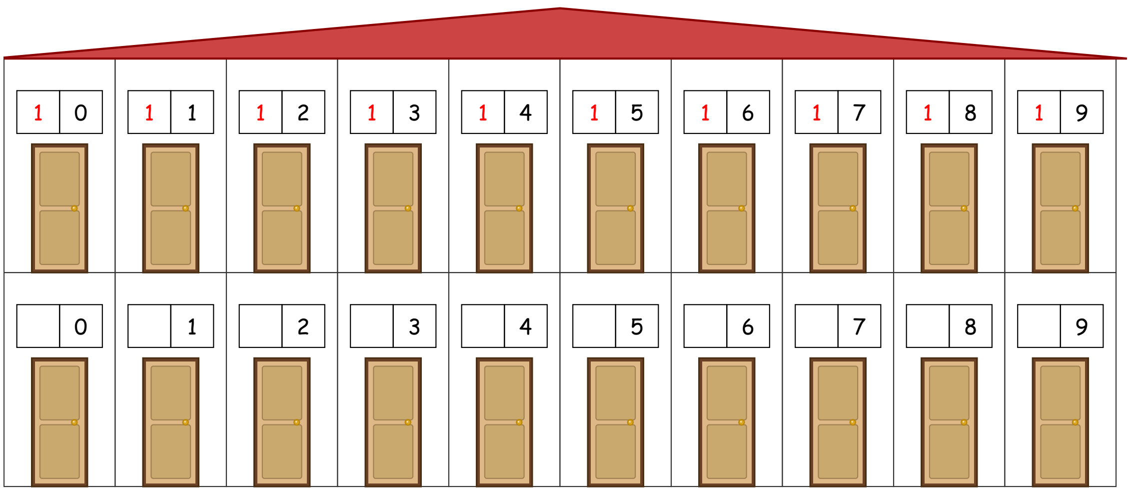


Figure 3.12: First 2 floors (floor 0 and floor 1) of our Grand Hotel.

To get from one room on the ground floor (floor 0) to the corresponding room on the first floor (floor 1), we have to climb a ladder.

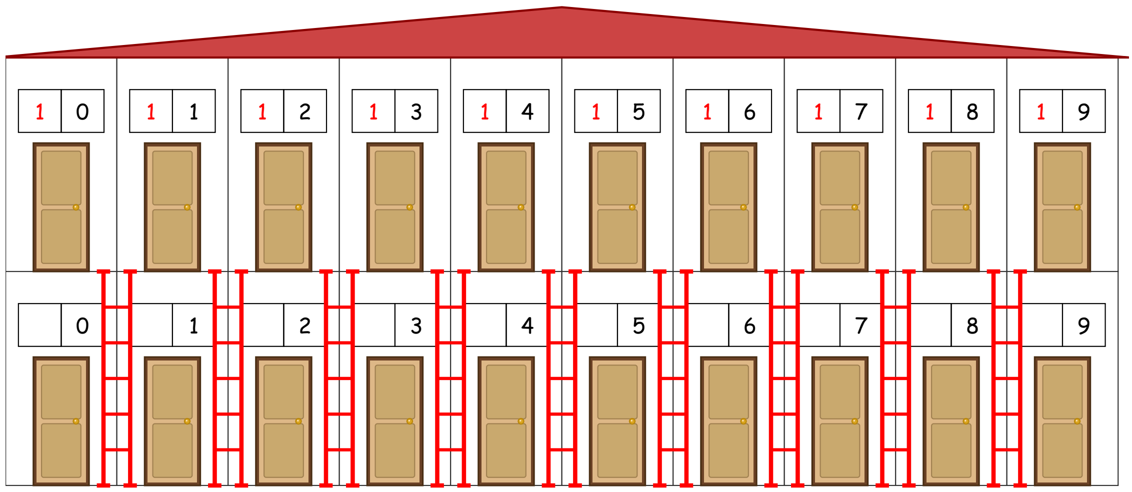


Figure 3.13: First 2 floors (floor 0 and floor 1) of our Grand Hotel with ladder.

Climbing a ladder means adding 10. For example, climbing a ladder from Room 7 on the ground floor will take you to Room 17 on the first floor, which is $17 = 7 + 10$.

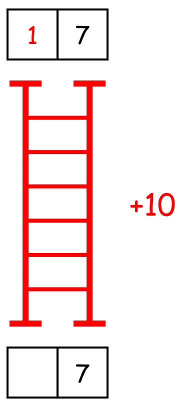


Figure 3.14: Climbing a ladder means adding 10.

If we need to go further, we have to climb more ladders. Think of the second placeholder in each room (the red digit) as the floor number or the number of ladders climbed from the ground floor to get there.

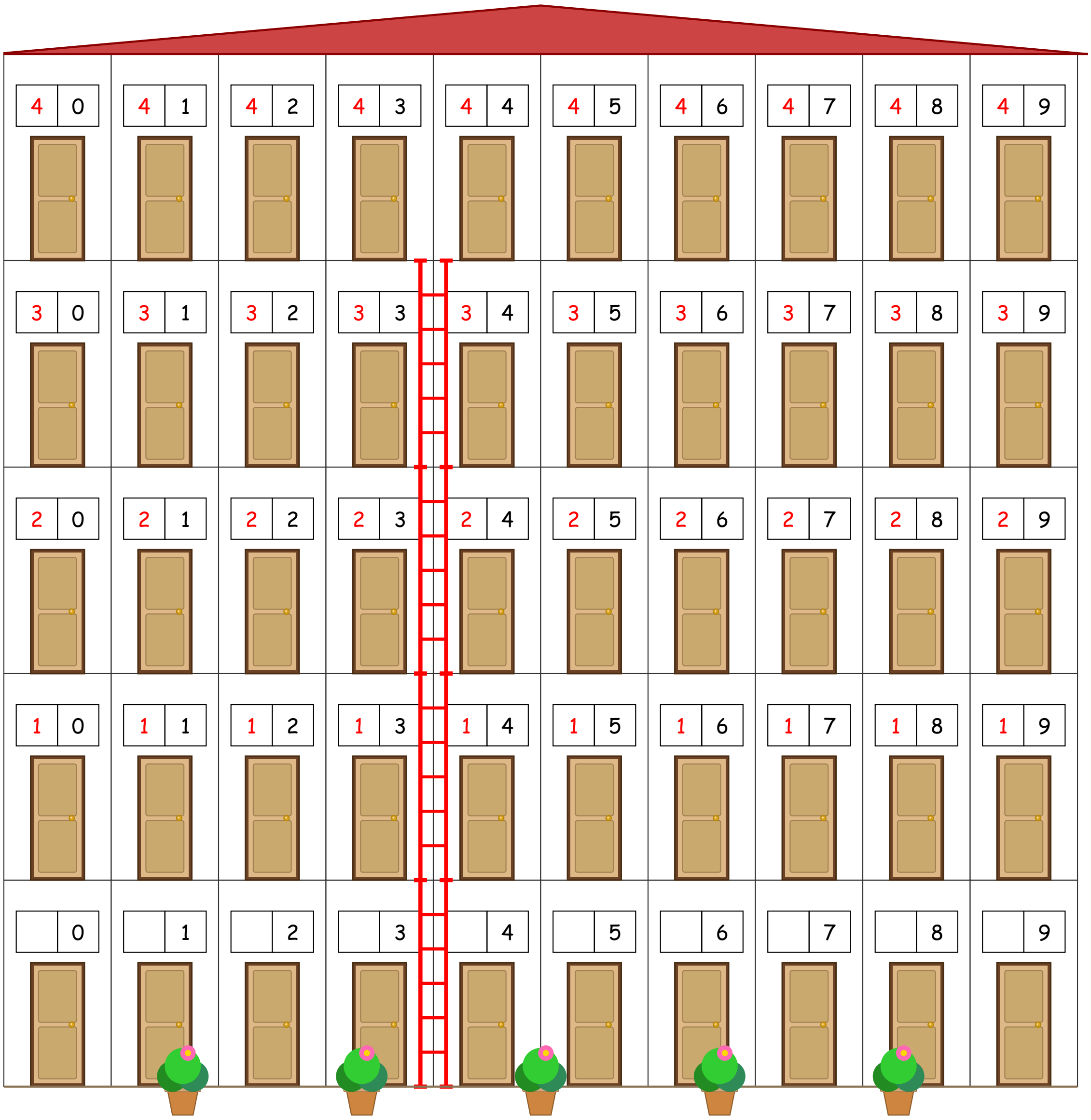


Figure 3.15: The first 5 floors of our Grand Hotel.

Ladders can be stacked. Every ladder represents the number of times we add 10.

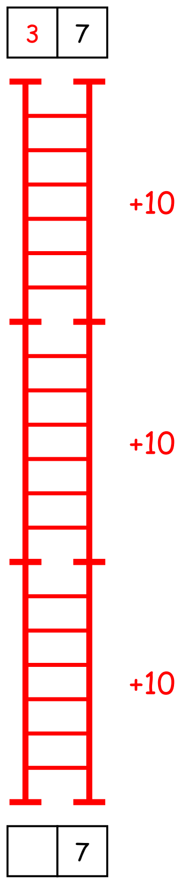


Figure 3.16: Climbing three ladders means adding $3 \times 10 = 10 + 10 + 10 = 30$.

Exercises

Fill in the blanks with the numbers of the missing rooms.

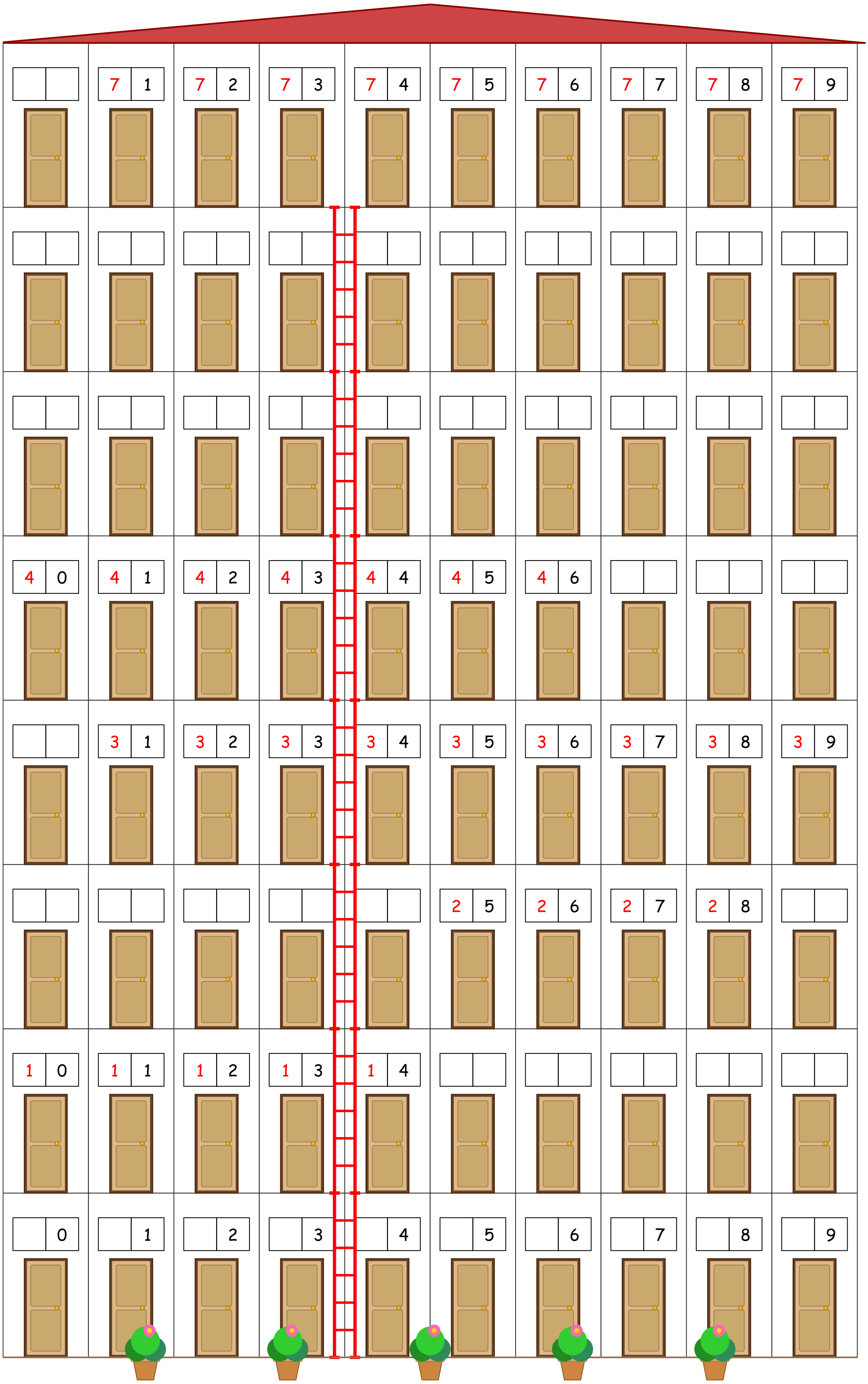


Figure 3.17: The first 5 floors of our Grand Hotel.

You saw that adding 10 is quite easy, you just have to climb one ladder to the next floor.

$$0 + 10 = 10$$

$$1 + 10 = 11$$

$$2 + 10 = \boxed{}$$

$$3 + 10 = \boxed{}$$

$$4 + 10 = \boxed{}$$

$$5 + 10 = \boxed{}$$

$$6 + 10 = \boxed{}$$

$$7 + 10 = \boxed{}$$

$$8 + 10 = \boxed{}$$

$$9 + 10 = \boxed{}$$

$$1 + 20 = 21$$

$$7 + 20 = \boxed{}$$

$$2 + 30 = \boxed{}$$

$$2 + 40 = \boxed{}$$

3.4 Base 10

10

Figure 3.18: The base of the decimal system is 10.

The reason the number system we have used so far is called the “decimal number system” and each floor of our Grand Hotel has ten rooms is because we are using a number system based precisely on the number 10. Ten is the base of this system. What does that mean?

On the one hand, it means that we have ten digits or symbols:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Figure 3.19: The decimal numbers. Ten digits.

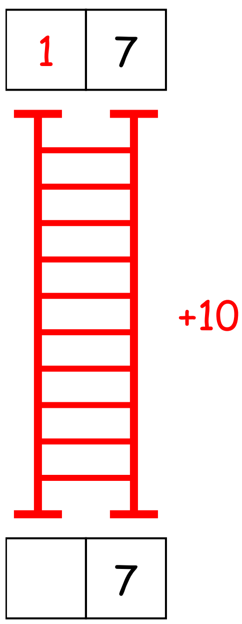


Figure 3.20: Climbing a ladder means adding 10.

It also means that all numbers greater than 9 are constructed adding one of these ten digits to combinations of 10[2].

In our Grand Hotel analogy, climbing a floor means adding up 10. Each time we climb a ladder, we add 10. If we climb two ladders we add two times 10, $2 \times 10 = 10 + 10 = 20$, if we climb 5 ladders we add five times 10, $5 \times 10 = 10 + 10 + 10 + 10 + 10 = 50$, and so on.

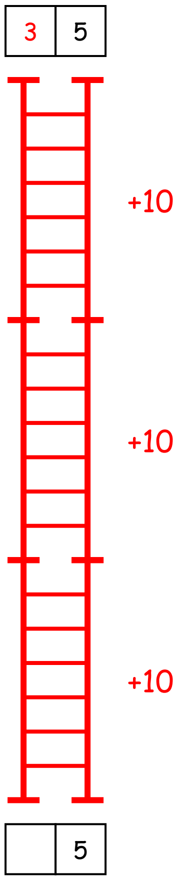


Figure 3.21: Climbing three ladders means adding $3 \times 10 = 10 + 10 + 10 = 30$.

This is how some of the new numbers we just learned are constructed using the base 10:

$$23 = 3 + (2 \times 10) = 3 + \underbrace{(10 + 10)}_{\text{2 times}} = 3 + 20$$

[2]Technically, powers of 10.

$$13 = 3 + (1 \times 10) = 3 + (\underbrace{10}_{1\text{ time}}) = 3 + 10$$

$$24 = 4 + (2 \times 10) = 4 + (\underbrace{10 + 10}_{2\text{ times}}) = 4 + 20$$

$$35 = 5 + (3 \times 10) = 5 + (\underbrace{10 + 10 + 10}_{3\text{ times}}) = 5 + 30$$

Exercises

Complete the following representations:

$$33 = 3 + (\boxed{3} \times 10) = 3 + \boxed{30}$$

$$76 = 6 + (\boxed{7} \times 10) = 6 + \boxed{70}$$

$$18 = \boxed{} + (\boxed{} \times 10) = \boxed{} + \boxed{}$$

$$19 = \boxed{} + (\boxed{} \times 10) = \boxed{} + \boxed{}$$

Same thing!

$$39 = \boxed{} + (\boxed{} \times 10) = \boxed{} + \boxed{}$$

$$21 = \boxed{} + (\boxed{} \times 10) = \boxed{} + \boxed{}$$

$$10 = \boxed{} + (\boxed{} \times 10) = \boxed{} + \boxed{}$$

$$11 = \boxed{} + (\boxed{} \times 10) = \boxed{} + \boxed{}$$

$$20 = \boxed{} + (\boxed{} \times 10) = \boxed{} + \boxed{}$$

Using base 10 also means that we can stack a maximum of 10 ladders of a given type. To continue climbing our hotel, we need to add a new type of ladder. Each time we add a position, we also add a new type of ladder. The numbers from right to left in the second position represent the number of ladders of the first type. The numbers in the third position represent the number of ladders of the second type, and so on.

Fill in the blanks with the numbers of the missing rooms in the upper floors.

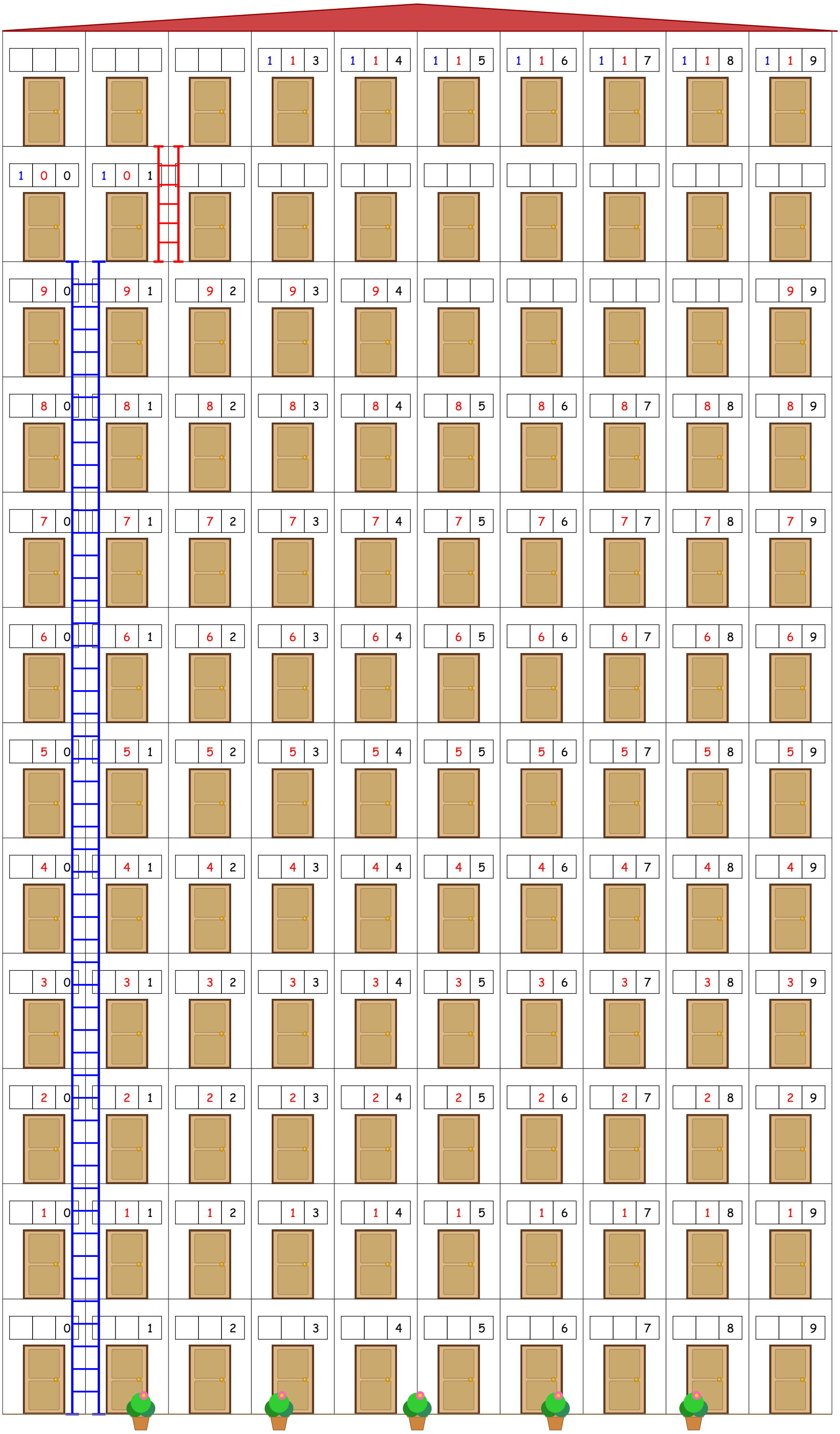


Figure 3.22: To reach rooms in floor number 11, for instance 112, we have to climb 1 blue ladder (equivalent to 10 red ladders) and 1 red ladder.

3.5 Comparing quantities

Let's start with a definition. The leftmost digit of a number is called the most significant digit because it contributes the most to the number's value.

Most significant digit

1000, 34, 4, 561, 333

Here are a few rules 👮‍♀️ to compare quantities of any number of digits.

- 👮‍♀️ If one number has more digits than the other, then it is larger.

1000 > 500 > 99 > 1

- 👮‍♀️ If two quantities have the same number of digits, we compare their most significant digits. The quantity with the greater most significant digit is greater.

90 > 70 > 50 > 30

- 👮‍♀️ If two quantities have the same number of digits and their most significant digits are the same, then we compare their second most significant digits. We continue this process until we break the tie or reach the least significant digit.

$57 > 53 > 51$

$539 > 532 > 530$

Exercises

Compare the following quantities and write $>$, $<$, or $=$ accordingly.

$1000 \;\square\; 999 \;\square\; 799 \;\square\; 590$

$100 \;\square\; 200$

$101 \;\square\; 100$

$4000 \;\square\; 3000 \;\square\; 2999 \;\square\; 1999$

650 □ 651 □ 652 □ 652

1234 □ 1234 □ 1234

18 □ 19 □ 20

4 Counting intermezzo

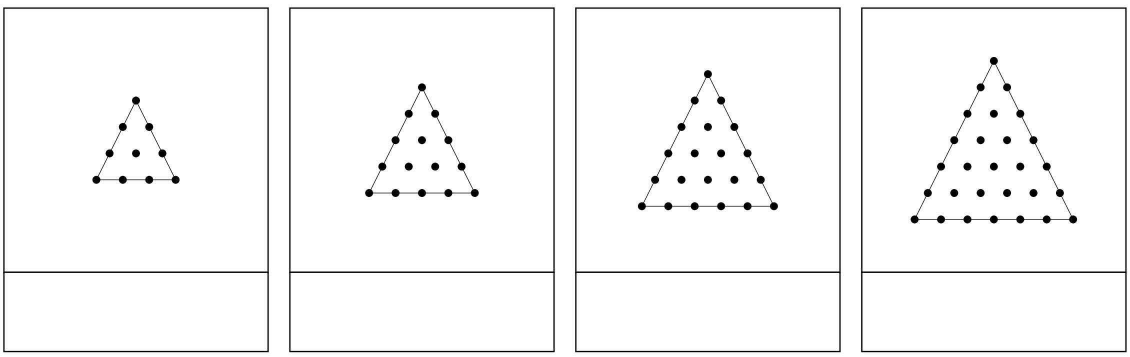
We have covered quite some ground. Before we continue, let's take a moment to recap and internalize what we have learned so far. Let's try out our new superpower: counting to the infinite and beyond.

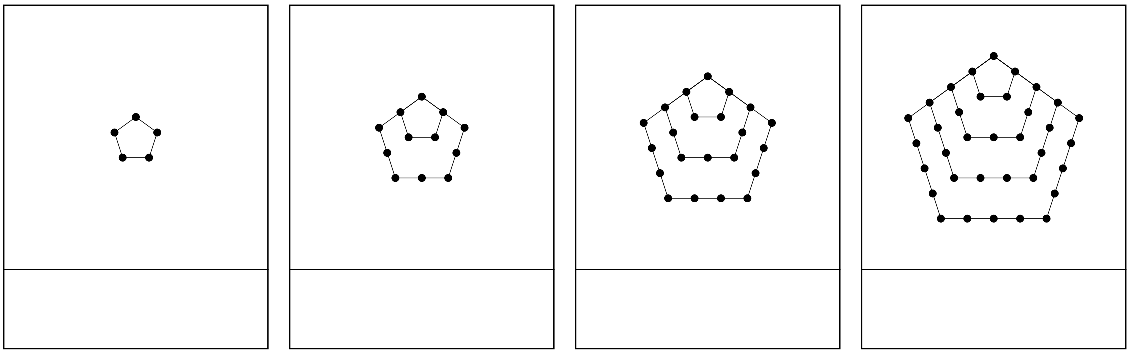
4.1 Counting objects

I have two goals for this section:

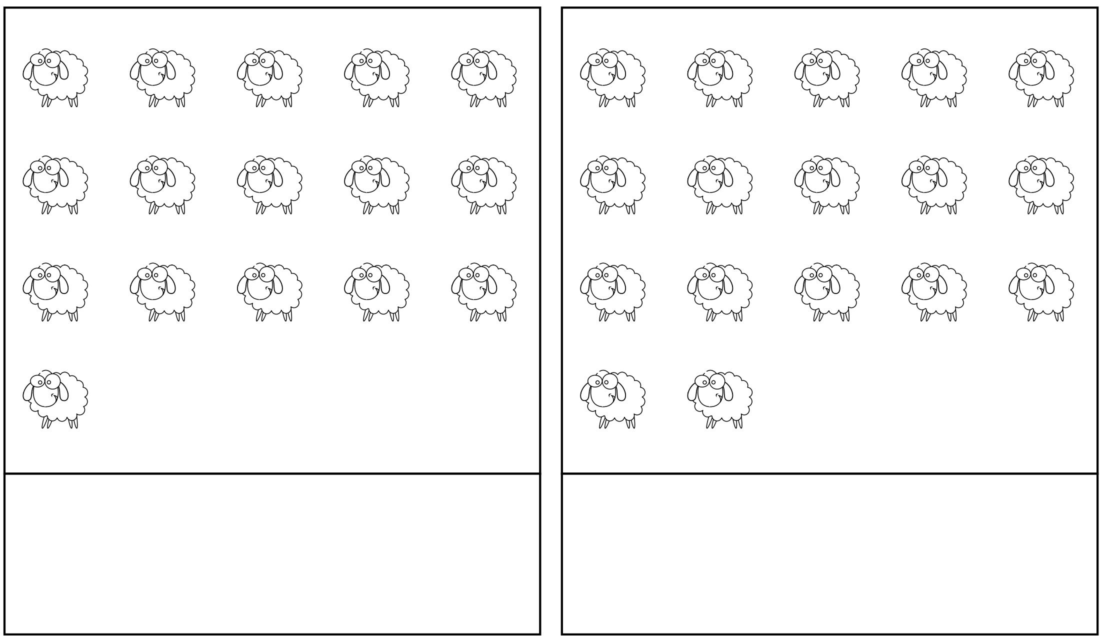
- First, I want you to get bored counting dots and sheeps so that later, when we learn to count faster, you will appreciate the power of multiplication and exponentiation.
- Second, I want you to intuitively figure out a faster way to count.

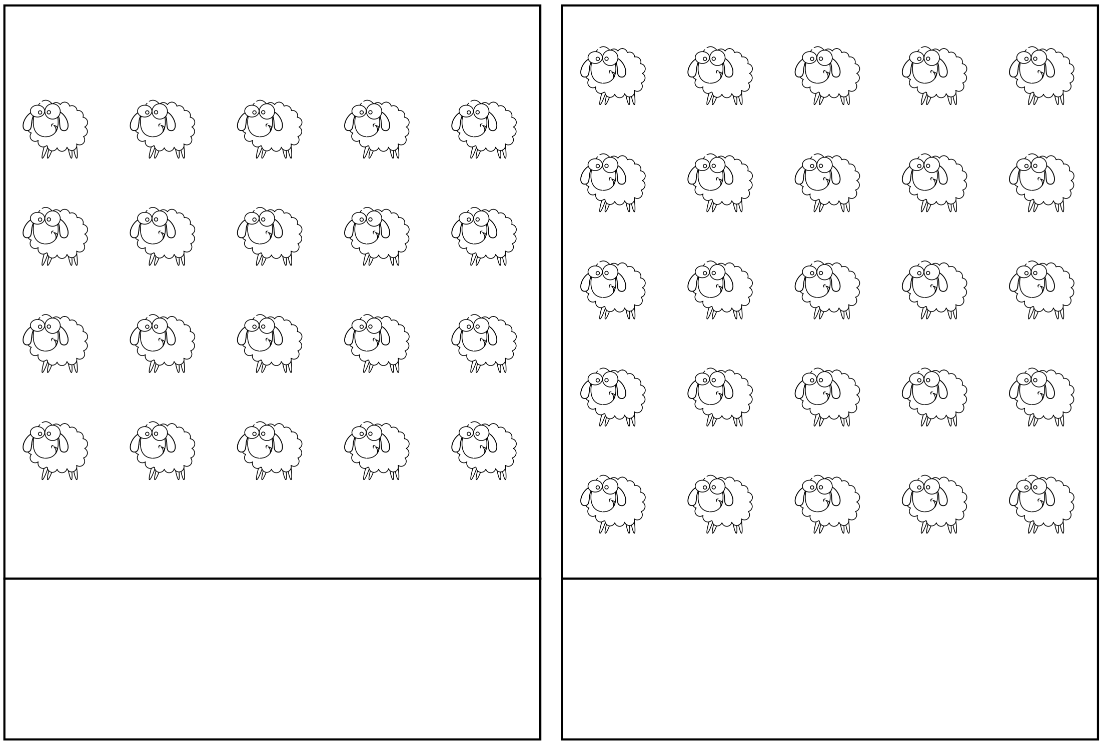
Identify the triangular and pentagonal numbers below. Remember, you need to count the number of points in each one.





Count the number of sheeps.





4.2 Magic squares

Magic squares are amazing mathematical objects that have fascinated people for centuries. In a square of size $n \times n$, the numbers from 1 to $n \times n$ are arranged so that the sum of each row, column, and diagonal is the same. For example, in this 3×3 magic square, the numbers from 1 to 9 are arranged so that the sum is always a constant. What is that constant? Add up every row, column, and diagonal to find out.

6	1	8
7	5	3
2	9	4

6	1	8
7	5	3
2	9	4

6	1	8
7	5	3
2	9	4

6	1	8
7	5	3
2	9	4

6	1	8
7	5	3
2	9	4

6	1	8
7	5	3
2	9	4

6	1	8
7	5	3
2	9	4

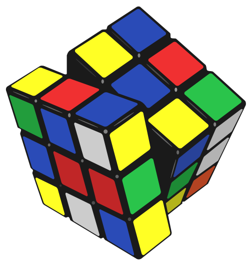
6	1	8
7	5	3
2	9	4

What about a magic square of size 4×4? What is the constant sum?

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

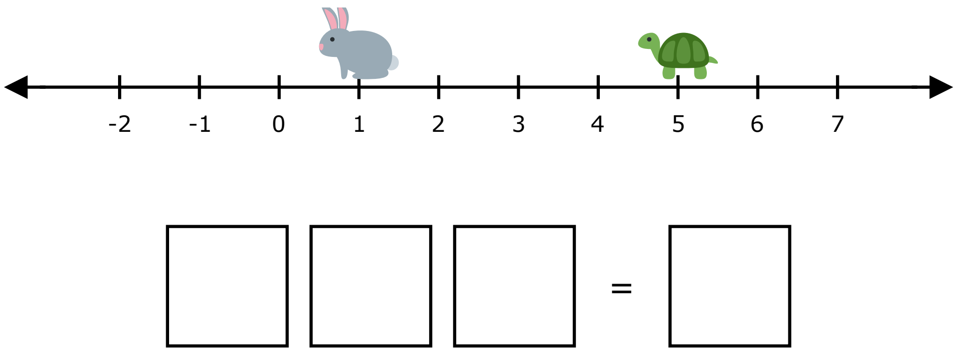
Magic squares and cubes

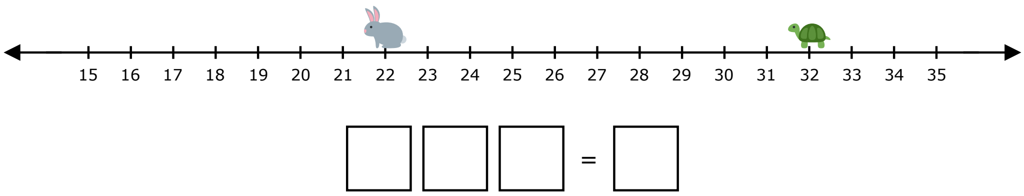
Did you know that magic squares can be constructed of any size except 2×2 and 6×6?
Also, Ernő Rubik, the creator of the Rubik's cube, initially called his amazing toy Bűvös Kocka, which means "magic cube" in Hungarian.

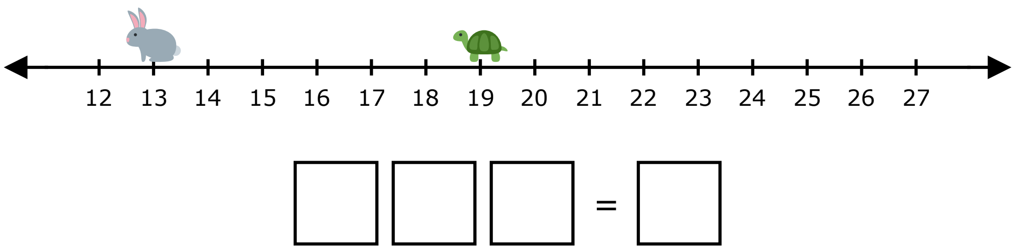


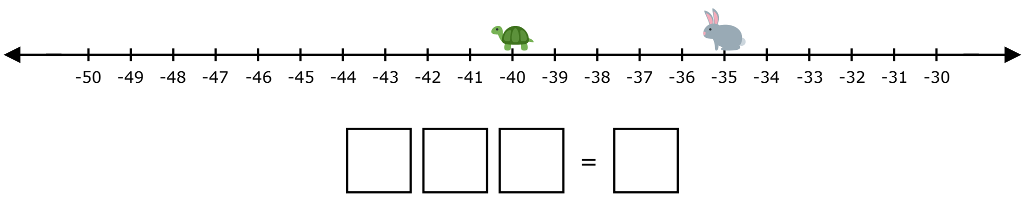
4.3 Number line

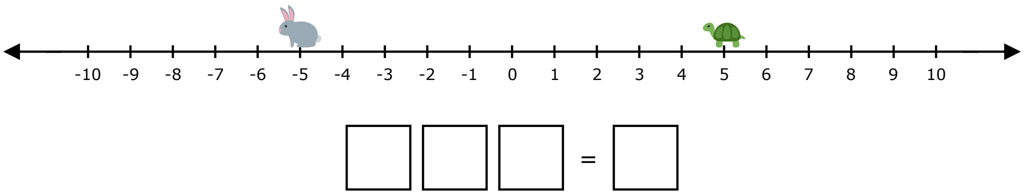
Find the distance between the  and the  on the number line.

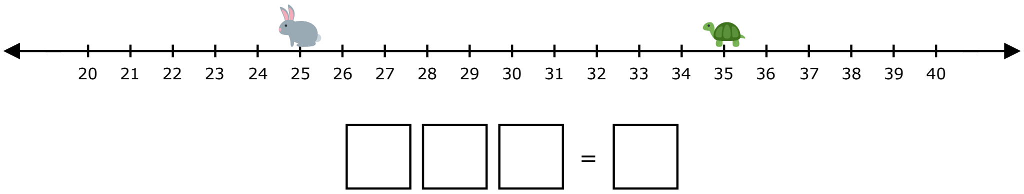

-2 -1 0 1 2 3 4 5 6 7
=

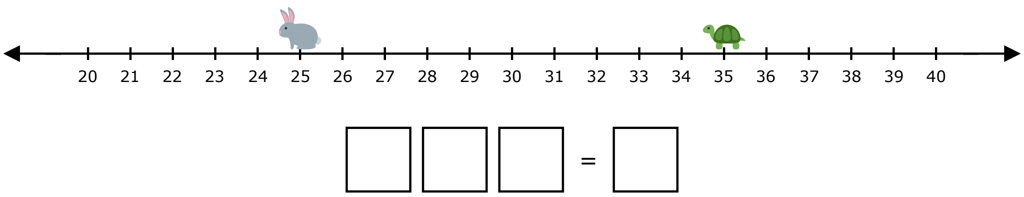

15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35
=


12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27
=


-50 -49 -48 -47 -46 -45 -44 -43 -42 -41 -40 -39 -38 -37 -36 -35 -34 -33 -32 -31 -30
=







4.4 Manhattan distance

If you live in or have visited a city with a grid-like structure, to get from one place to another that are not on the same street, you need to walk along two perpendicular streets. This distance is called the Manhattan distance.

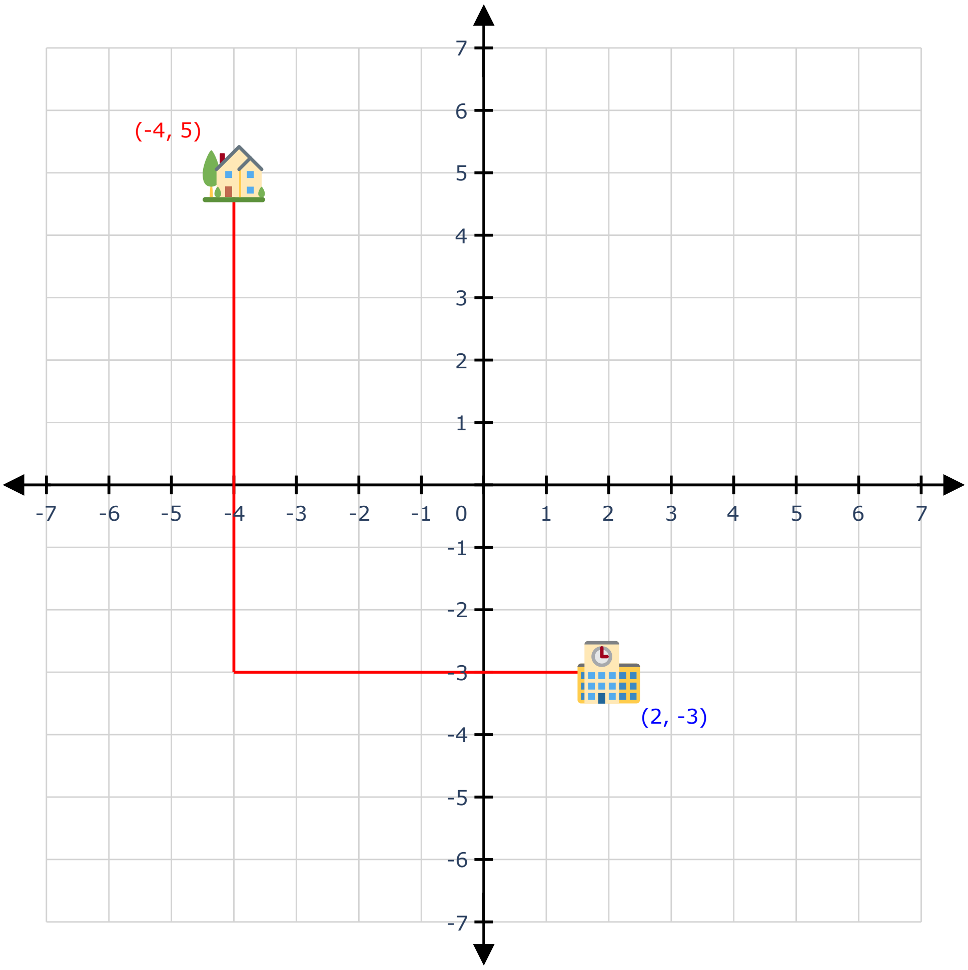


Figure 4.1: Manhattan distance between two points on the Cartesian plane.

$$\underbrace{\text{Manhattan distance}}_{d_M} = \underbrace{\text{horizontal distance}}_{d_h} + \underbrace{\text{vertical distance}}_{d_v}$$

Since addition is commutative, and distances are always positive, the order in which you walk, either horizontally or vertically first, doesn't matter. The total distance traveled will be the same either way.

For example, the Manhattan distance between the  and the  is the same in the following figures, even though you walk in a different order in each figure. In the first figure, you walk horizontally first and then vertically, while in the second figure, you walk vertically first and then horizontally.

$$d_M(\text{🐢}, \text{🐇}) = \underbrace{d_h(\text{🐢}, \text{🐇})}_{2-(-4)=2+4=6} + \underbrace{d_v(\text{🐢}, \text{🐇})}_{2-(-2)=2+2=4} = 10$$

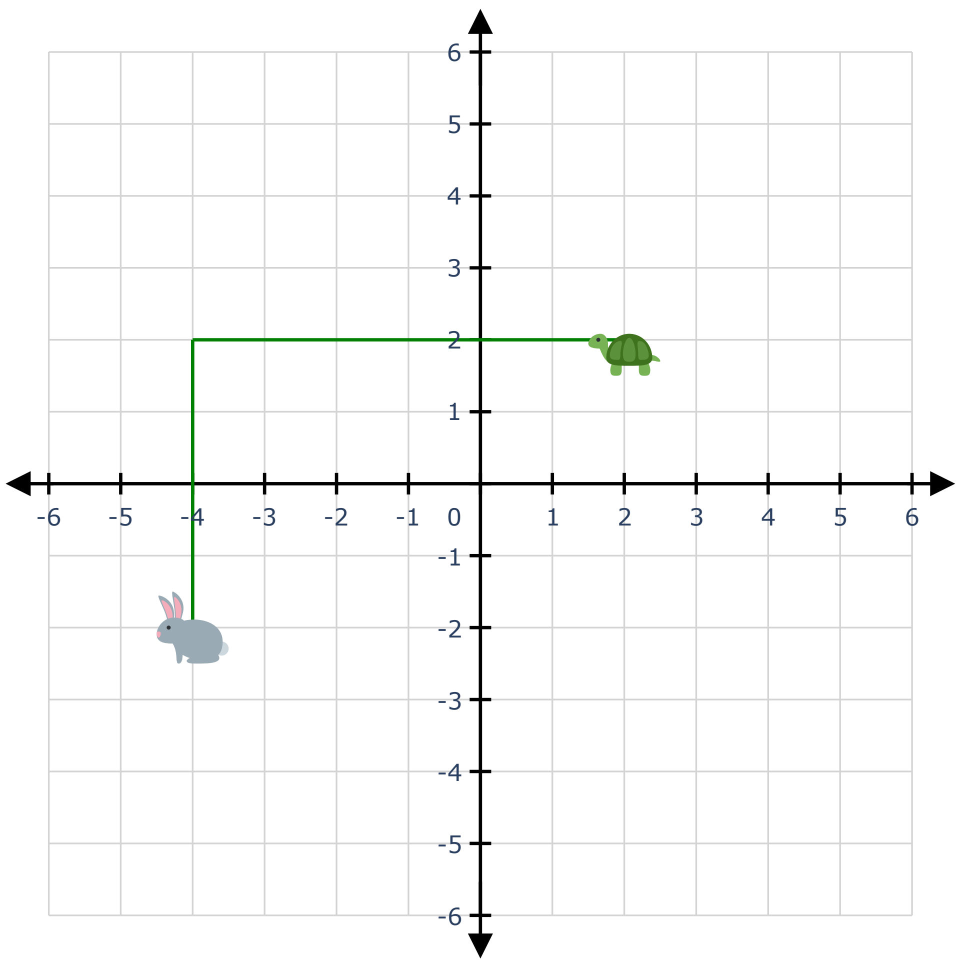


Figure 4.2: Walking first horizontally and then vertically.

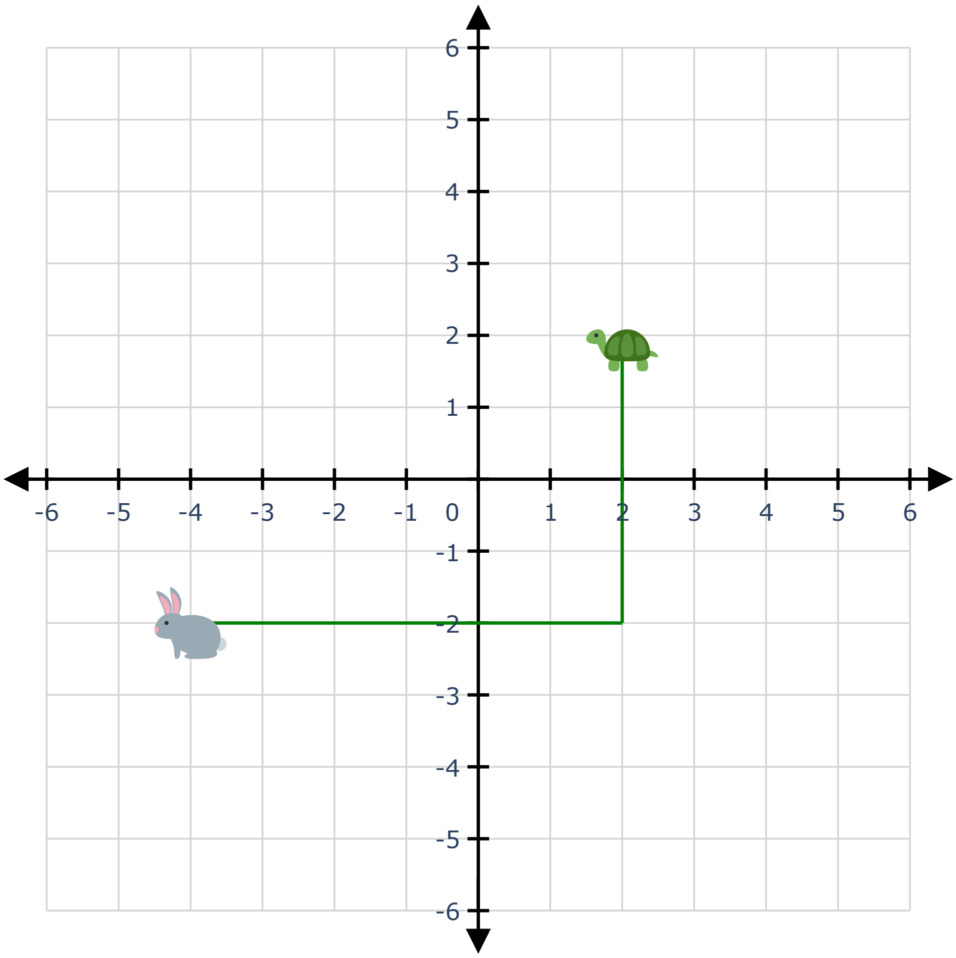


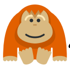
Figure 4.3: Walking first vertically and then horizontally.

The following is a map of the city. Your house  is located at (-7, -9).

- Find the following distances:

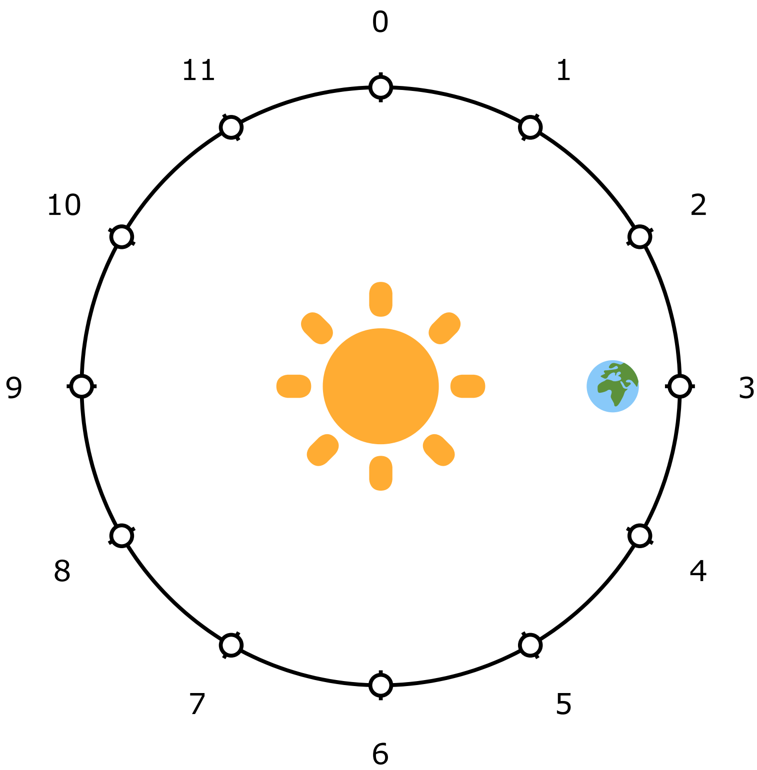
$$d_M(🏡, 🏫) = \square + \square = \square$$
$$d_M(🚉, 🦧) = \square + \square = \square$$
$$d_M(🚉, 🚒) = \square + \square = \square$$

- Which is farther from your house: the school 🏫 or the amusement park 🎠?

- If you are at the fire station , which is closer to you: the school  or the zoo ?
- If it takes you 15 minutes to walk from home to school, how long would it take you to walk back home, assuming you walk at the same speed? Why?

4.5 Clock arithmetic

Consider the group $\mathbb{Z}_{12}$ with elements $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ and addition modulo 12.



Complete the group table for addition in $\mathbb{Z}_{12}$:

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1		2	3	4	5	6	7	8	9	10	11	0
2												
3												
4												
5												
6												
7												
8												
9												
10												
11												

4.6 Is $\mathbb{N}$ a group?

We saw that $\mathbb{Z}$, $\mathbb{Z}_6$, and $\mathbb{Z}_{12}$ are examples of groups. However, the natural numbers $\mathbb{N}$ are not a group under addition. Why?

Hint: Think of the three things we verified to determine if a set and an operation form a group. Which one of those three conditions is not met?

- ? **Closure and associativity:** Is the sum of two natural numbers always a natural number? Is the order in which you group numbers when adding them does not change the result?
- ? **Identity:** Is there a natural number that, when added to any other natural number, results in the same number?

- **Inverses:** For every natural number, is there another natural number that, when added to itself, yields the identity element?

4.7 Comparing quantities

- Which is faster: the plane or the bullet train?
- What is faster: the Tour de France champion or a taxi?
- Which is faster: the comet or the rocket?
- Which is the fastest?
- Which is the slowest?

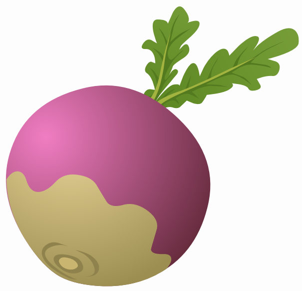
5 Counting faster

5.1 Kangaroo math 🦘

The Gigantic Turnip

“The Gigantic Turnip” is a cumulative or *chain* Russian folk tale. In a cumulative tale, the action repeats and builds in a cumulative manner.
The story is about an old man and an old woman who lived in a crooked cottage and owned the following animals:

- 🐄
- 🐖🐖
- 🐈‍⬛🐈‍⬛🐈‍⬛
- 🐔🐔🐔🐔
- 🦢🦢🦢🦢🦢
- 🐤🐤🐤🐤🐤🐤

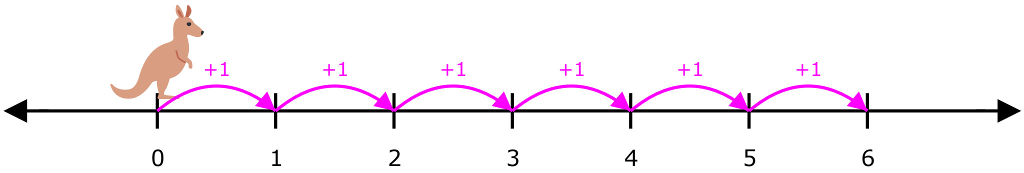


Read more about this and other cumulative tales at https://wikipedia.org/

The numbers representing the number of each of these animals form an arithmetic progression, or arithmetic sequence, because the difference between any two consecutive numbers is constant. In this case, the common difference is 1.

$$1,\ 2,\ 3,\ 4,\ 5,\ 6$$

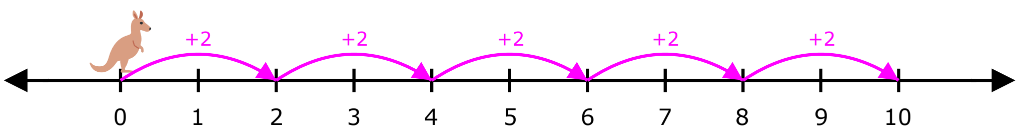
In fact, the natural numbers $\mathbb{N}$ are an arithmetic progression as well. As we saw before, the natural numbers can be generated by starting from 0 and repeatedly adding 1.



Our kangaroo  generates the natural numbers by jumping by ones.

0, 1, 2, 3, 4, 5, 6,
7, 8, 9, 10, 11, 12, ...

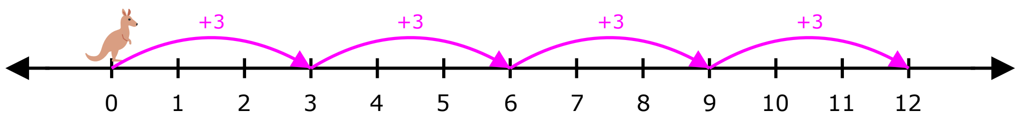
There are two things you should know about our kangaroo . First, his name is Joey, and he can jump more than one unit at a time. For example, he can jump two units at a time.



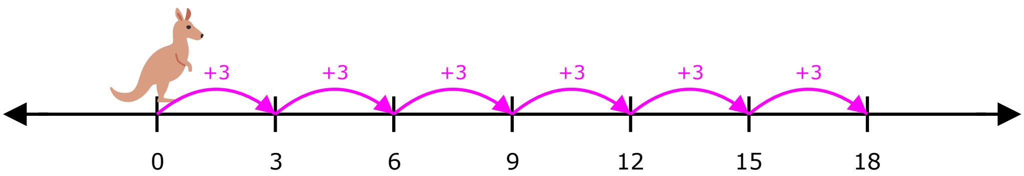
If we follow Joey, we get the sequence:

0, 2, 4, 6, 8, 10, 12, 14, 16, 18, ...

Joey can, of course, also jump in threes.



On the number line, we can skip the numbers that aren't part of Joey's jumps. The sequence remains the same; we are simply changing how we visualize it.



In both cases, the resulting sequence is the same:

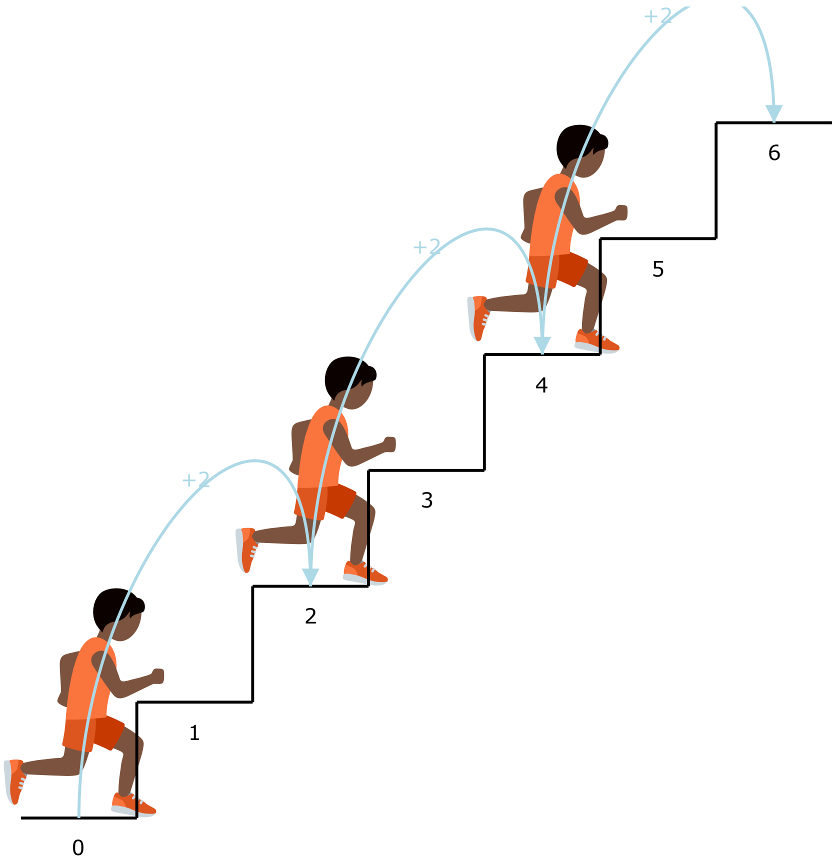
0, 3, 6, 9, 12, 15, 18, 21, 24, ...

Why is Kangaroo math important?

There are two important reasons to learn Kangaroo Math. First, it allows you to count faster. Second, it provides a solid foundation for learning multiplication, division, and exponentiation, as we will see in the next sections.

Counting by jumping two, three, or five units at a time allows you to count faster, just as climbing a staircase becomes quicker when you skip steps.

For example, you can climb a staircase twice as fast if you jump two steps at a time instead of one step at a time.

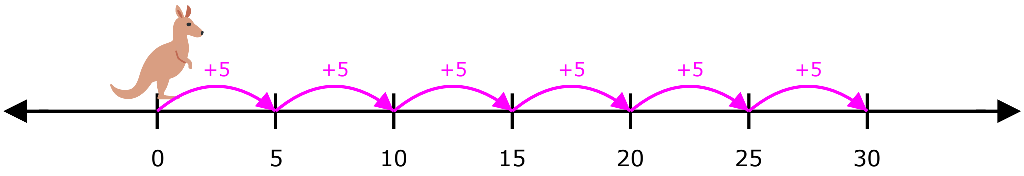


Kangaroo math and multiplication

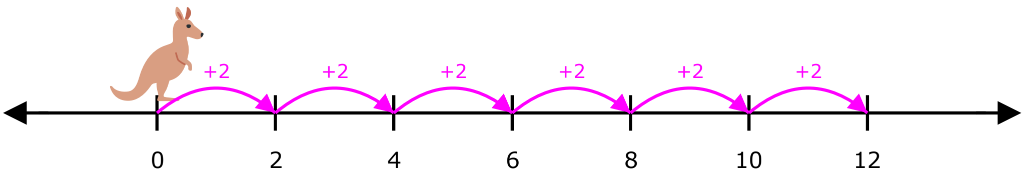
Multiplication is simply a shortcut for counting faster. We will learn later that there are even faster counting tricks.

Exercises

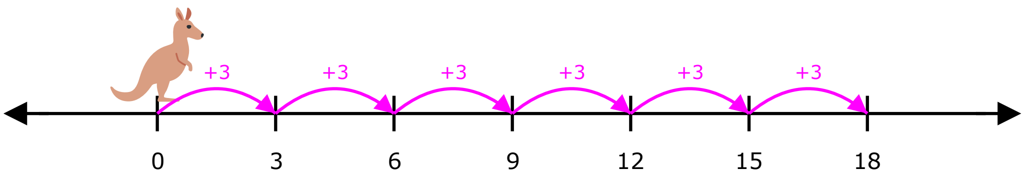
Let's practice our Kangaroo Math! Write down the sequences generated when Joey jumps n units at a time.



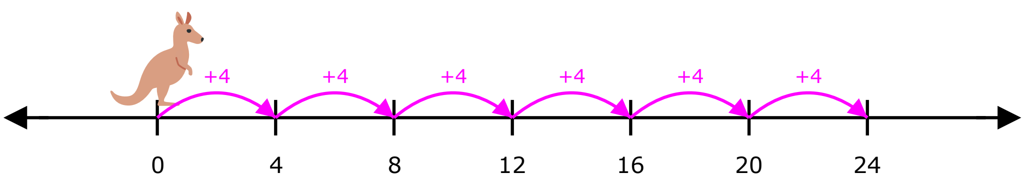
0, 5, 10, 15, 20, 25, 30,
35, 40, 45, 50, 55, 60, 65,
70, 75, 80, 85, 90, 95, ...



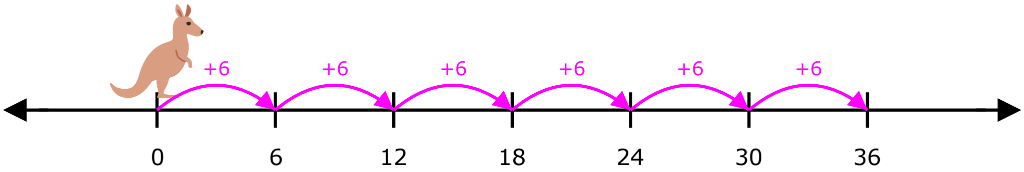
0, 2, 4



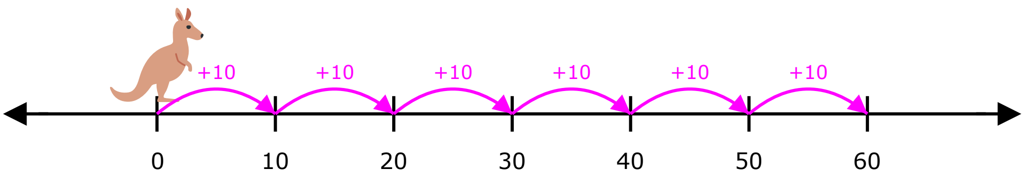
0, 3, 6



0, 4, 8

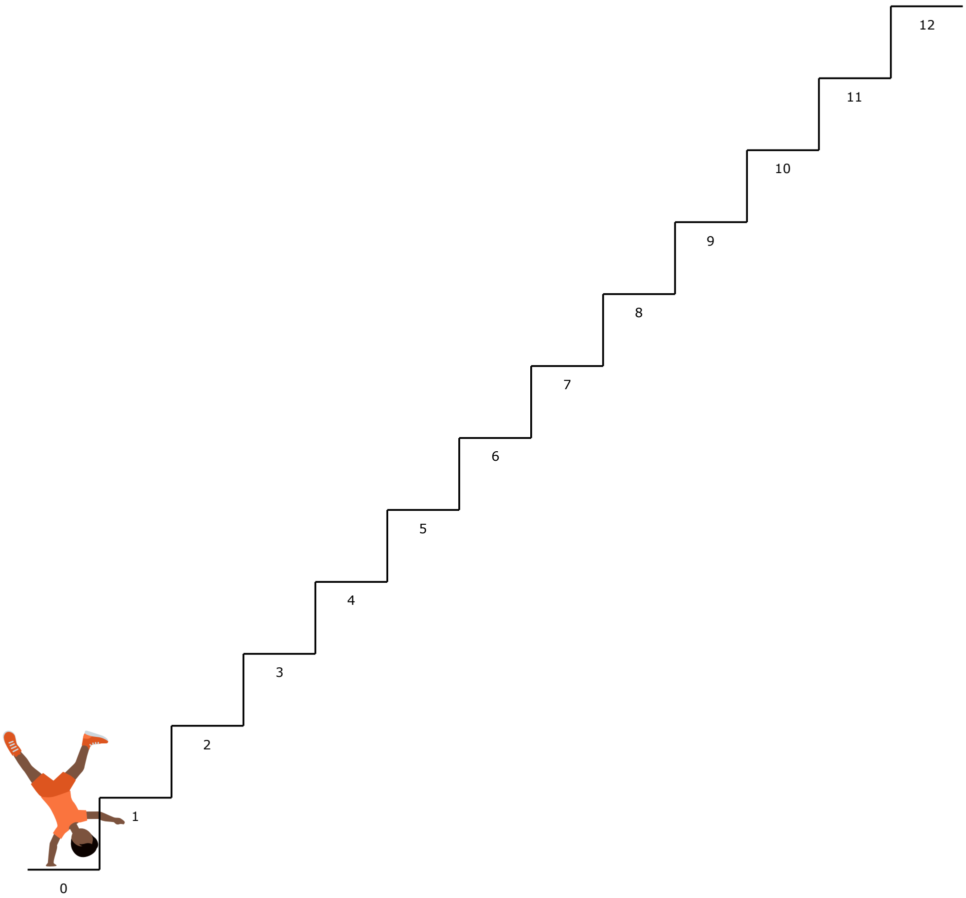


0, 6, 12



0, 10, 20

Based on the following image, answer the following questions:



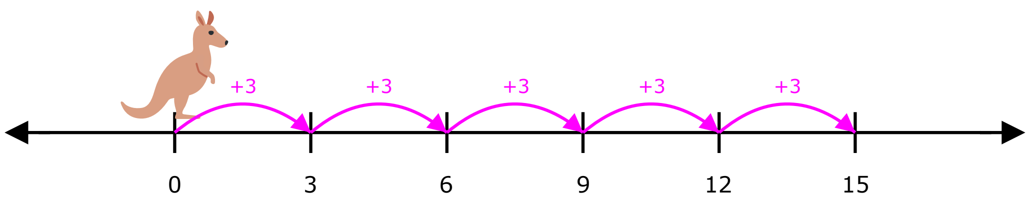
- When you climb the stairs one step at a time, it takes you 12 seconds to reach the top. If you jump 3 steps at a time at the same speed, how long would it take you to get to the top?
- If you want to climb the stairs in 2 seconds, how many steps would you have to jump at a time?

5.2 Spotting patterns

As we saw earlier, multiplication is repeated addition—the special case that results from adding a value to itself multiple times. For example:

$$\underbrace{3+3+3+3+3}_{5\text{ times}} = 5 \times 3$$

Adding 3 to itself five times is the same as counting by threes five times.



That's the connection between multiplication and Kangaroo Math!

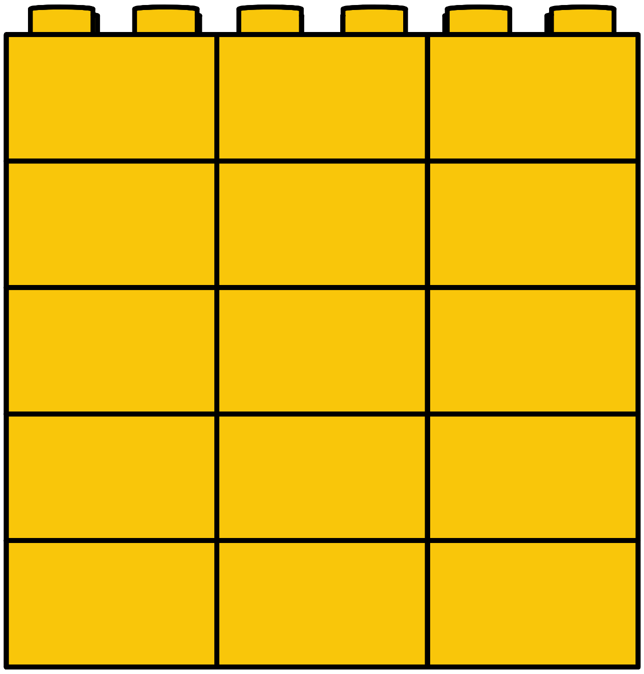
To count faster, we must learn to do two related yet independent things.

1. Spot a pattern in the objects we want to count.
2. Practice and memorize—or rather, internalize—the results of common multiplications.

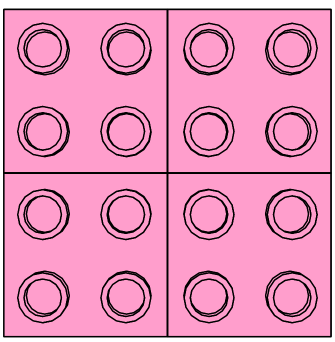
In the next section, we will start learning how to master multiplication results. For now, focus on spotting patterns in the objects you want to count.

Exercises

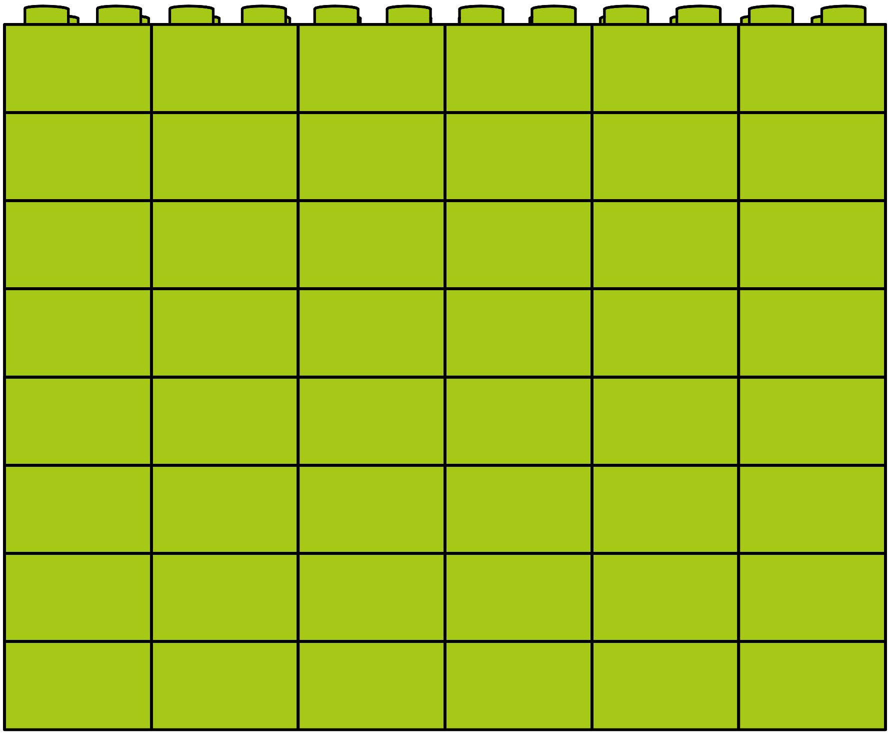
Try to identify the pattern in the following quantities, then write it down as a multiplication.



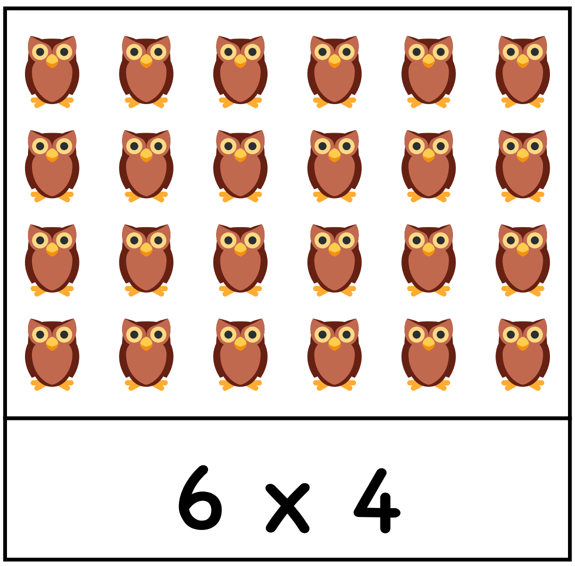
Here we have 5 rows of 3 bricks each. The total number of bricks is then 5×3.

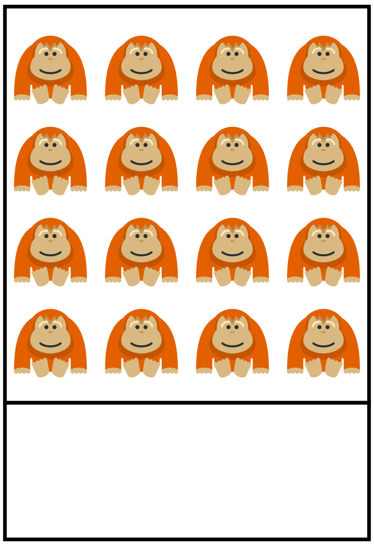


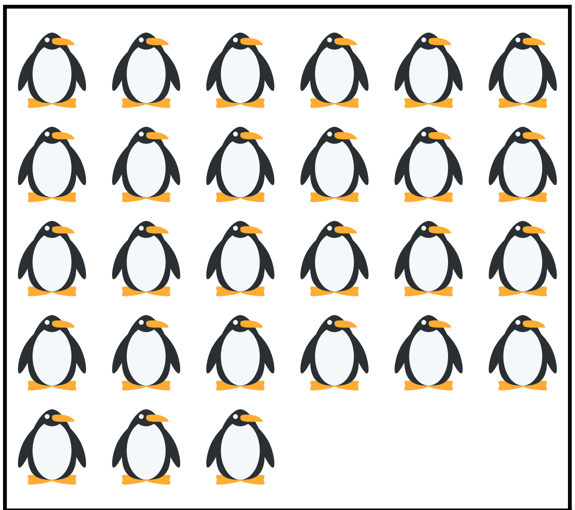
Here the total number of pink bricks is $\boxed{} \times \boxed{}$.



Here the total number of green bricks is $\square \times \square$.

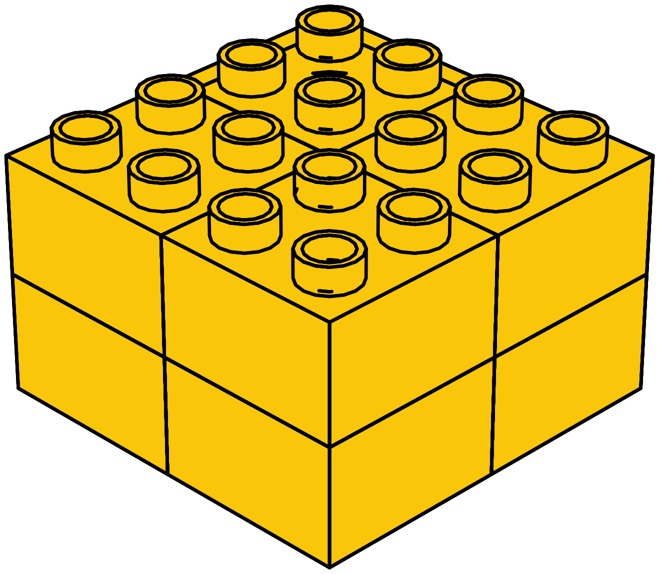




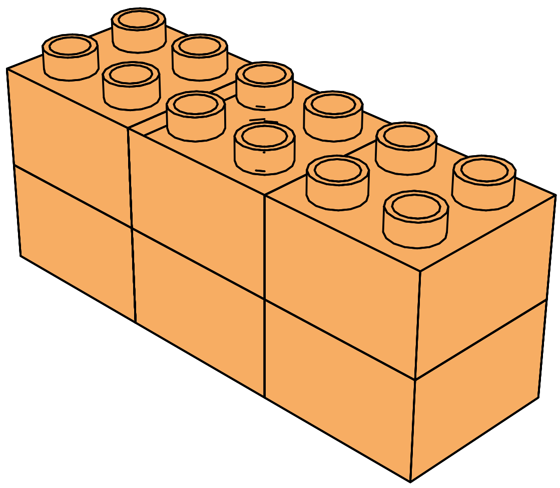


$$6 \times 4 + 3$$

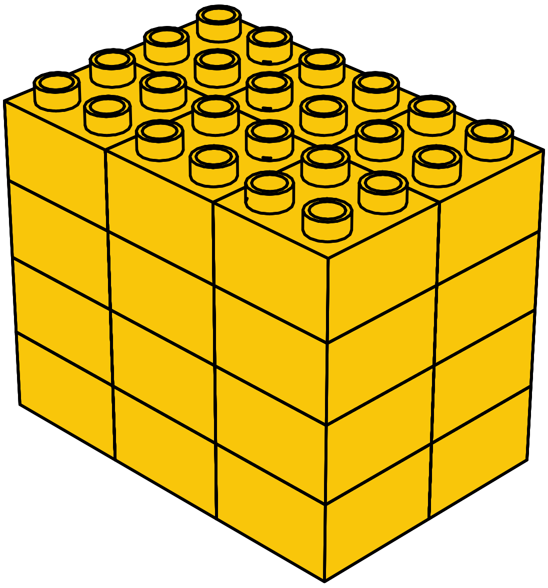
There are 6 rows of 4 penguins each, plus an extra row with 3 penguins. Note that the last row cannot be included in the multiplication because it has a different number of penguins than the other rows.



There are 2 layers of 2 rows of 2 bricks each. The total number of bricks is then $2 \times 2 \times 2$.



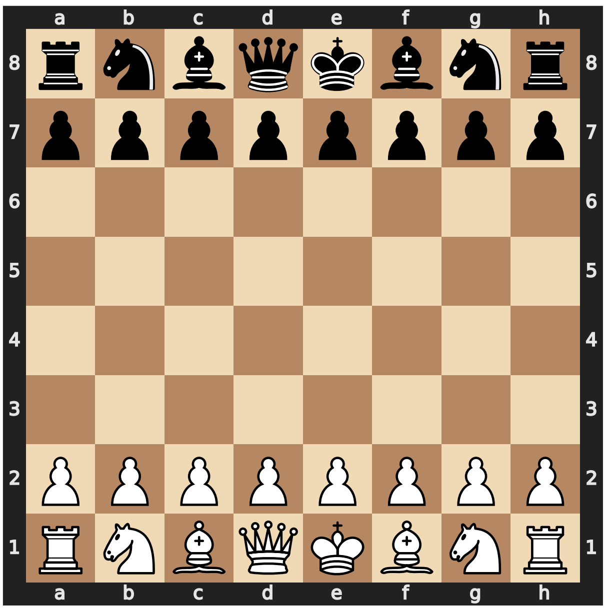
The total number of orange bricks is $\square \times \square \times \square$.



The total number of yellow bricks is $\square \times \square \times \square$.

Based on the chessboard below, answer the following questions. Can you express your answers using multiplication?

- How many squares are there in total on the chessboard?
- How many white pieces are on the chessboard?
- How many pieces are there in total (white and black)?
- How many squares are empty?



5.3 Multiplication tables

The Arabian Nights of Numbers

Beremiz Samir is a gifted mathematician who can count faster than anyone else. He can look at a flock of birds or a herd of camels and instantly tell you how many there are. His mathematical genius guides him through exciting and dangerous adventures as he solves puzzles and riddles, helps people, and finds love. Read more about Beremiz Samir in The Man Who Counted by Malba Tahan.

We can also learn to count quickly. As we saw in the previous section, the first step is identifying patterns. The second step is to practice and memorize the results of basic multiplication. Mastering these calculations will give you superpowers and lay the groundwork for algorithms for multiplying larger numbers.

We will use tables like the one below to practice our multiplication skills. Don't worry about memorizing them; with practice, you will get better.

These tables are very similar to the ones we saw before for addition when defining groups.

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15
4	0	4	8	12	16	20
5	0	5	10	15	20	25

The values in the table are the *products* of multiplying the row and column numbers. For example, to find the result of 3×4, first locate the row with the number 3, then the column with the number 4. The cell where the row and column intersect contains the product, which is 12 in this case. Therefore, $3 \times 4 = 12$.

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15
4	0	4	8	12	16	20
5	0	5	10	15	20	25

In the same way, we can compute for instance 2×5.

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15
4	0	4	8	12	16	20
5	0	5	10	15	20	25

If you look closely, you will see that the results of previous multiplications appear twice in the table.

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15
4	0	4	8	12	16	20
5	0	5	10	15	20	25

> **Why is that?**
>
> Stop and think!

Those values appear twice because multiplication, like addition, is *commutative.*

$$a \times b = b \times a$$

In previous examples, we saw that $2 \times 5 = 5 \times 2$ and $3 \times 4 = 4 \times 3$. This property makes the multiplication table symmetric with respect to the diagonal.

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15
4	0	4	8	12	16	20
5	0	5	10	15	20	25

What does it mean? It means good news! We only need to learn one half of the table: either the upper or lower triangle. Hooray! 🥳

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1		1	2	3	4	5
2			4	6	8	10
3				9	12	15
4					16	20
5						25

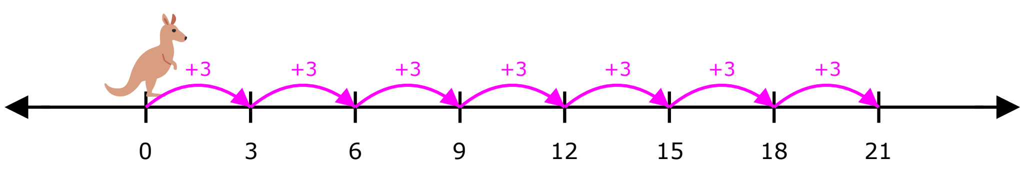
However, keep in mind that the product you want to calculate may not be directly accessible, so you may need to use the commutative property to find it. For instance, if you want to calculate 5×2, you could look for 2×5 instead.

What are the sequences of each row and column?

By looking at one row or column in particular, such as the sequence of values in row n, which is the same as the sequence of values in column n, we obtain an arithmetic progression of multiples of n. For example, the sequence of values in row 3 is the same as the sequence of values in column 3. We obtain the sequence $0, 3, 6, 9, 12, 15, 18, \ldots$.

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	8	10	12	14
3	0	3	6	9	12	15	18	21
4	0	4	8	12	16	20	24	28
5	0	5	10	15	20	25	30	35
6	0	6	12	18	24	30	36	42
7	0	7	14	21	28	35	42	49

In other words, the values in a given row or column correspond to the series in Kangaroo Math when Joey jumps n units at a time.



What are the values in the diagonal?

> **Question**
>
> What do the values in the diagonal of the multiplication table represent?

×	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1		1	2	3	4	5	6	7	8	9	10
2			4	6	8	10	12	14	16	18	20
3				9	12	15	18	21	24	27	30
4					16	20	24	28	32	36	40
5						25	30	35	40	45	50
6							36	42	48	54	60
7								49	56	63	70
8									64	72	80
9										81	90
10											100

These are the results of multiplying a number by itself. We will soon learn that this operation is called *squaring* a number, and the result is called the *square* of the number.

$$\underbrace{a \times a}_{\boxed{2\ \text{times}}} = a^{\boxed{2}}$$

Number of times we multiply a by itself.

For instance:

$$1 \times 1 = 1^2 = 1$$

$$2 \times 2 = 2^2 = 4$$

$$3 \times 3 = 3^2 = 9$$

$$4 \times 4 = 4^2 = 16$$

$$5 \times 5 = 5^2 = 25$$

$$6 \times 6 = 6^2 = 36$$

$$7 \times 7 = 7^2 = 49$$

$$8 \times 8 = 8^2 = 64$$

$$9 \times 9 = 9^2 = 81$$

$$10 \times 10 = 10^{2} = 100$$

$$\vdots$$

Exercises

Based on this multiplication table:

×	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1		1	2	3	4	5	6	7	8	9	10	11	12
2			4	6	8	10	12	14	16	18	20	22	24
3				9	12	15	18	21	24	27	30	33	36
4					16	20	24	28	32	36	40	44	48
5						25	30	35	40	45	50	55	60
6							36	42	48	54	60	66	72
7								49	56	63	70	77	84
8									64	72	80	88	96
9										81	90	99	108
10											100	110	120
11												121	132
12													144

Calculate the following products:

$$2 \times 7 = \boxed{}$$

$$7 \times 2 = \boxed{}$$

$$11 \times 11 = \boxed{}$$

$$12 \times 12 = \boxed{}$$

$$5 \times 9 = \boxed{}$$

$$7 \times 6 = \boxed{}$$

$$11 \times 1 = \boxed{}$$

$$7 \times 1 = \boxed{}$$

$$8 \times 0 = \boxed{}$$

$$12 \times 0 = \boxed{}$$

$$11^2 = \boxed{}$$

$$12^2 = \boxed{}$$

$$10^2 = \boxed{}$$

$$4^2 = \boxed{}$$

- What happens when you multiply a number by 0?
- What happens when you multiply a number by 1?
- Write down the arithmetic series generated by counting in sevens.

0, 7, 14, ...

The following are are the rules for multiplying negative numbers:

- If the signs are the same (both positive or both negative), the result is positive. For example, $(-2) \times (-3) = 6$ and $2 \times 3 = 6$.
- If the signs are different (one positive and one negative), the result is negative. For example, $(-2) \times 3 = -6$ and $2 \times (-3) = -6$.

In other words, the sign is the exclusive or (XOR), or the modulo 2 sum of the signs of the factors.

Question

Why multiplying two negative numbers results in a positive number?

$$2 \times -7 = \boxed{}$$

$$7 \times -2 = \boxed{}$$

$$-11 \times -11 = \boxed{}$$

$$12 \times 12 = \boxed{}$$

$$5 \times -9 = \boxed{}$$

$$7 \times -6 = \boxed{}$$

$$11 \times -1 = \boxed{}$$

$$-7 \times -1 = \boxed{}$$

$$2 \times 0 = \boxed{}$$

$$-2 \times 0 = \boxed{}$$

$$(-10)^2 = \boxed{}$$

$$(-4)^2 = \boxed{}$$

5.4 Exponentiation

As we saw earlier, repeated addition is multiplication. For example, 4×7 is equivalent to adding 7 four times: $7 + 7 + 7 + 7$. In the general case, we have:

$$\underbrace{a + a + \cdots + a}_{n \text{ times}} = n \times a$$

Because mathematics is recursive, much like the Gigantic Turnip, we can now ask ourselves the following question:

> **Question**
>
> If repeated addition is multiplication, then what is repeated multiplication?

As we've seen, multiplying a number by itself is called squaring the number. We symbolize this with a cute little superscript.

Number of times we multiply a by itself.

$$a \times a = a^{\boxed{2}}$$

This little number represents how many times we multiply a number by itself. It is called the **exponent**.

$$\underbrace{a \times a \times \cdots \times a}_{n \text{ times}} = a^n$$

Just as we call it "squaring" when the exponent is two, we call it "cubing" when the exponent is three because we are defining the volume of a cube.

Here are some examples of exponentiation:

$$
\begin{aligned}
2^2 &= 2 \times 2 &= 4 \\
2^3 &= 2 \times 2 \times 2 &= 8 \\
4^2 &= 4 \times 4 &= 16 \\
3^3 &= 3 \times 3 \times 3 &= 27
\end{aligned}
$$

5.5 Exponentiation of 10

A special case of exponentiation occurs in the decimal system when the base is 10.

$$
\begin{aligned}
10^1 &= 10 &= 10 \\
10^2 &= 10 \times 10 &= 100 \\
10^3 &= 10 \times 10 \times 10 &= 1000 \\
10^4 &= 10 \times 10 \times 10 \times 10 &= 10000
\end{aligned}
$$

In general we have:

$$\underbrace{10 \times 10 \times \cdots \times 10}_{n \text{ times}} = 10^{n} = 1\underbrace{000 \cdots 000}$$

Number of zeros after the 1

Below is a list of the first powers of 10:

$$10^{0} = 1$$
0 zeros

$$10^{1} = 10$$
1 zero

$$10^{2} = 100$$
2 zeros

$$10^{3} = 1000$$
3 zeros

$$10^{4} = 10000$$
4 zeros

$$10^{5} = 100000$$
5 zeros

Notice that 10^0 is equal to 1. This is true for any number, not just 10. $a^0 = 1$.

As I mentioned before, mathematics is recursive.

> **Question**
>
> In the same way that repeated addition is multiplication and repeated multiplication is exponentiation, what is repeated exponentiation?

We will stop at exponentiation. As an interesting fact, repeated exponentiation is called tetration, which allows us to build towers of numbers like this:

$$2^{3^{3^3}}$$

Exercises

Calculate the following powers:

$$2^3 = \boxed{}$$

$$2^4 = \boxed{}$$

$$3^2 = \boxed{}$$

$$10^0 = \boxed{}$$

$$10^1 = \boxed{}$$

$$10^5 = \boxed{}$$

$$10^4 = \boxed{}$$

5.6 Positional number system revisited

Just as calculating powers of 10 is easy, so too is multiplying a digit by a power of 10.

$$3 \times 10^2 = 3 \times 100 = 300$$

$$7 \times 10^3 = 7 \times 1000 = 7000$$

$$2 \times 10^4 = 2 \times 10000 = 20000$$

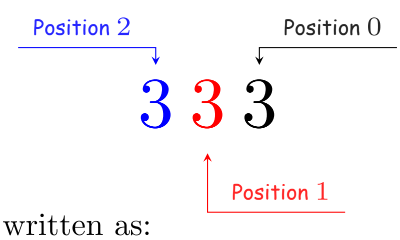
$$3 \times 10^5 = 3 \times 100000 = 300000$$

Do you see the pattern?

Now that we have this information, we can finally formalize the positional number system.

Remember that in a positional number system, the value of a digit depends on its position.

> **Question**
>
> But what is the value of each digit?



The same number can be written as:

$$\underset{}{3}\;3\;3 = 3 \times 10^{\boxed{2}} + 3 \times 10^{\boxed{1}} + 3 \times 10^{\boxed{0}}$$

Position 2 Position 1 Position 0

Therefore, we can write the same number as the sum below:

$$\begin{array}{r} 300 \\ +30 \\ +3 \\ \hline 333 \end{array}$$

Here is another example:

$$4\;1\;0\;1 = 4 \times 10^{\boxed{3}} + 1 \times 10^{\boxed{2}} + 0 \times 10^{\boxed{1}} + 1 \times 10^{\boxed{0}}$$

Position 3 Position 2 Position 1 Position 0

$$
\begin{array}{r}
\color{green}{4}000 \\
+\color{blue}{1}00 \\
+\color{red}{0}0 \\
+1 \\
\hline
\color{green}{4}\color{blue}{1}\color{red}{0}1
\end{array}
$$

In general, the value of each digit d_i is given by:

$$d_i \times 10^{p_i}$$

Where p_i is the position of the digit d_i in the number, starting at 0 for the rightmost digit.

> **Fun Fact**
>
> This holds true for any base, not just 10, and can be extended to non-integer numbers as well.

Exercises

Write each of the following numbers as its constituent parts:

$$123 = 100 + 20 + 3$$

$$100 = 100 + 0 + 0$$

$$4567 = 4000 + 500 + 60 + 7$$

$$17 = \boxed{} + \boxed{}$$

$$23 = \boxed{} + \boxed{}$$

$$1001 = \boxed{} + \boxed{} + \boxed{} + \boxed{}$$

$$1000 = \boxed{} + \boxed{} + \boxed{} + \boxed{}$$

$$555 = \boxed{} + \boxed{} + \boxed{}$$

$$171 = \boxed{} + \boxed{} + \boxed{}$$

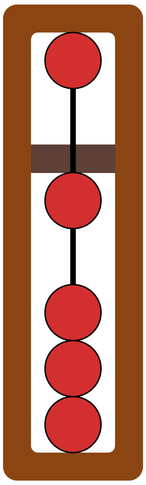
$$79 = \boxed{} + \boxed{}$$

$$42 = \boxed{} + \boxed{}$$

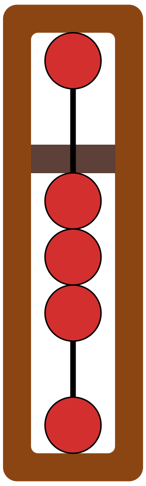
$$10381 = \boxed{} + \boxed{} + \boxed{} + \boxed{} + \boxed{}$$

We saw in section Section 1.8 how to represent values using a single-column Soroban abacus.

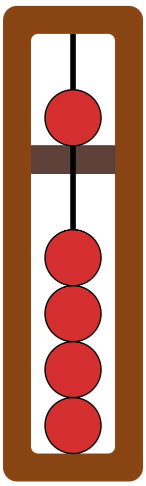
Here are some examples to refresh your mamory:



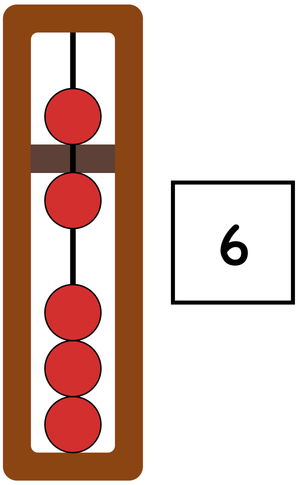
1



3

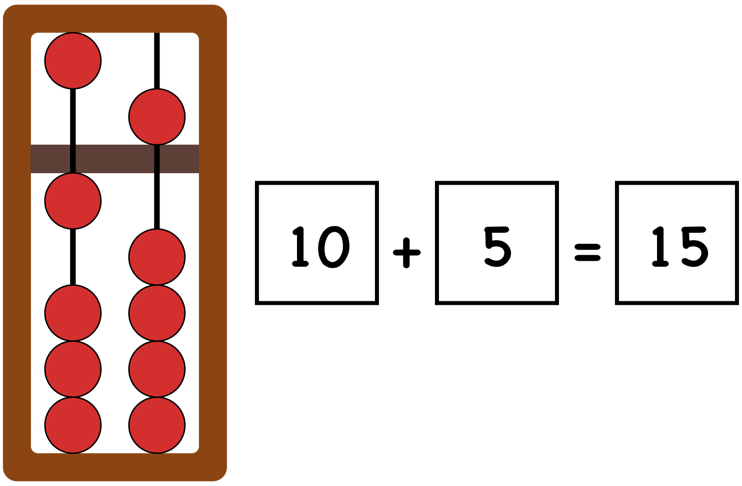


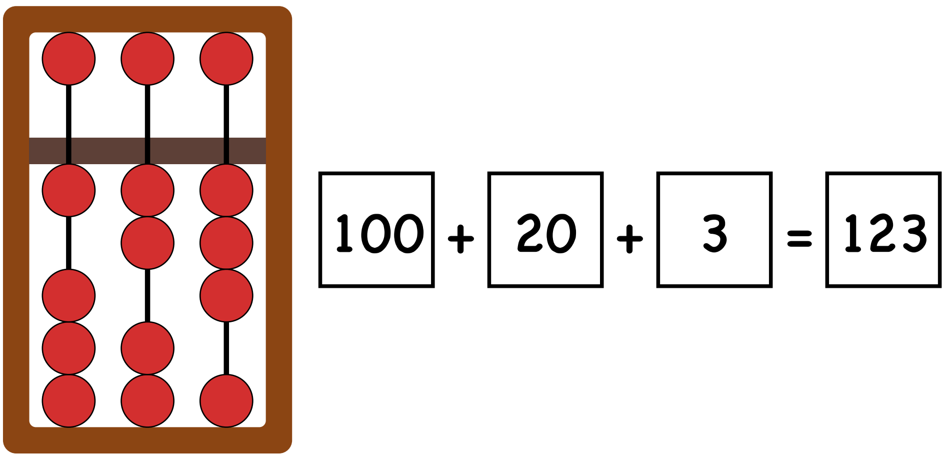
5

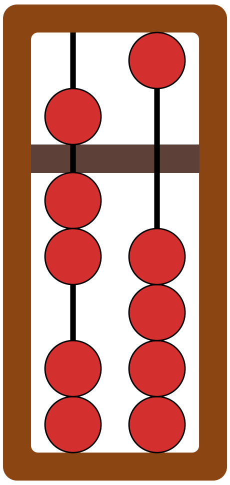


The Soroban abacus follows the same logic as the positional number system, meaning that the first column to the right represents units, that is the value multiplied by 10^0, the next column represents tens, that is the value multiplied by 10^1, the next column represents hundreds, that is the value multiplied by 10^2, and so on.

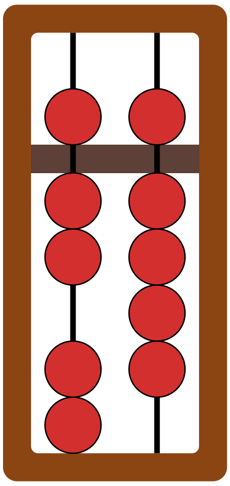
Complete the following representations:



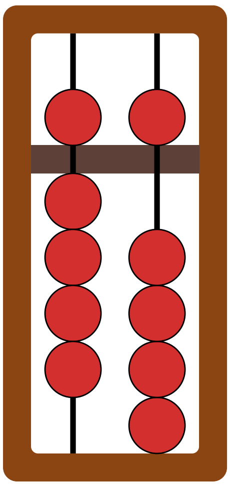




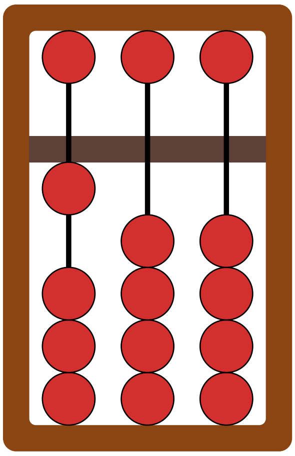
$\square + \square = \square$



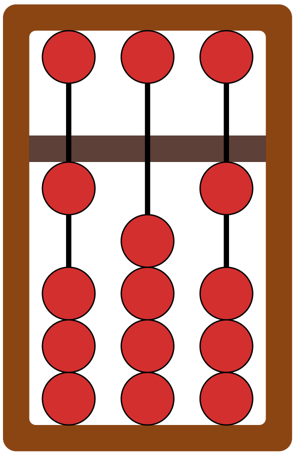
$\square + \square = \square$



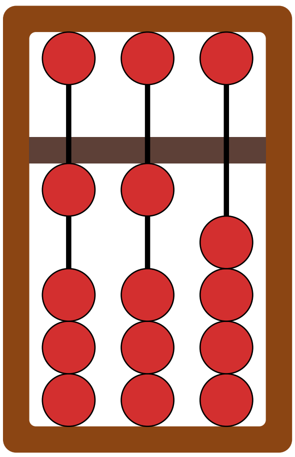
$\square + \square = \square$



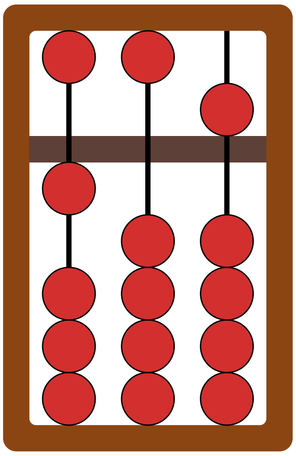
$\square + \square + \square = \square$



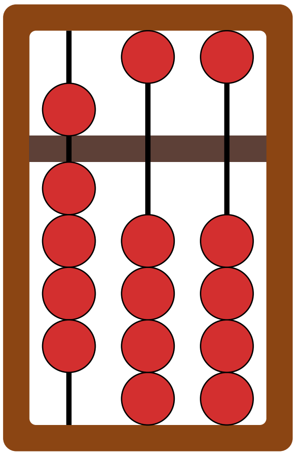
$\square + \square + \square = \square$



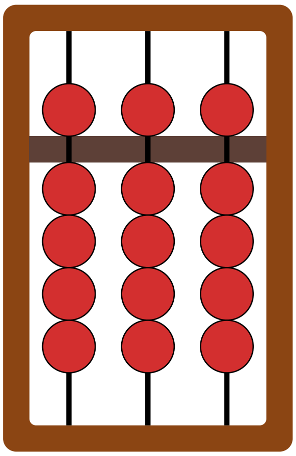
$\square + \square + \square = \square$



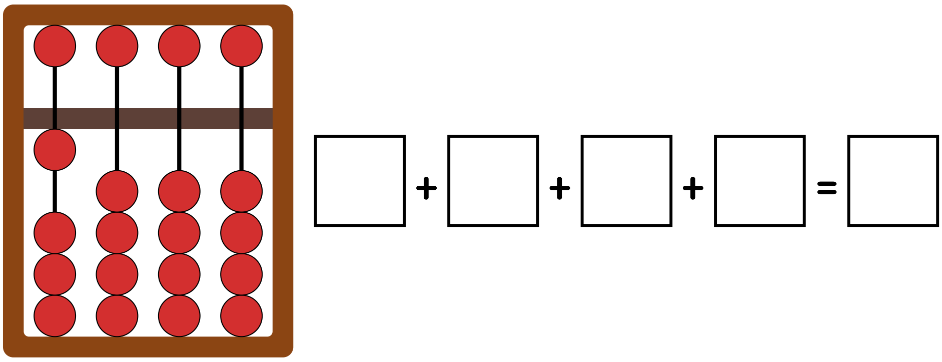
$\square + \square + \square = \square$

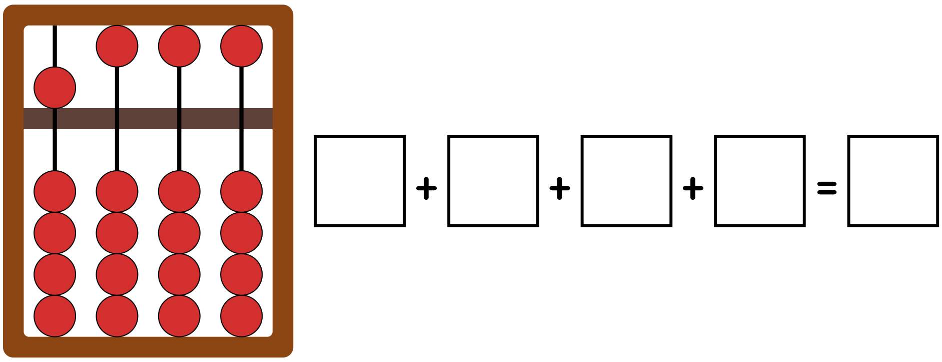


$\square + \square + \square = \square$



$\square + \square + \square = \square$

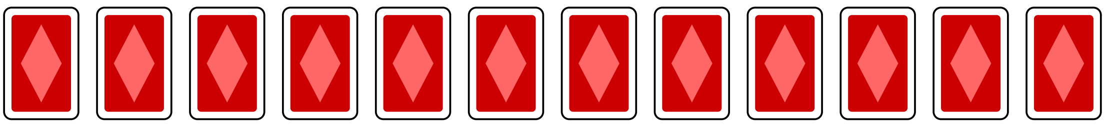


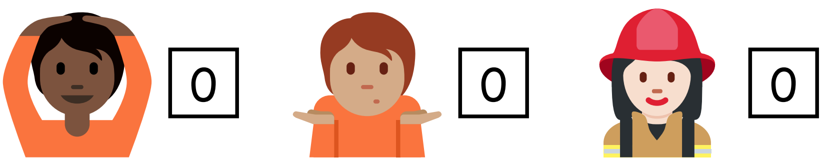


5.7 Division

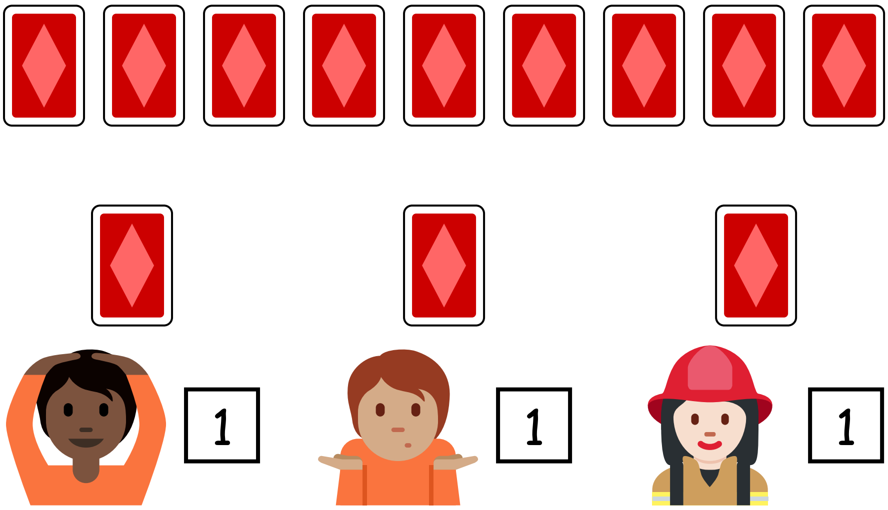
Imagine that you are playing a card game with your friends. It could be Uno, The Mind, or any other card game in which each player starts with an equal number of cards.

Let's say the game has 12 cards and 3 are three players. Initially, each player has 0 cards.

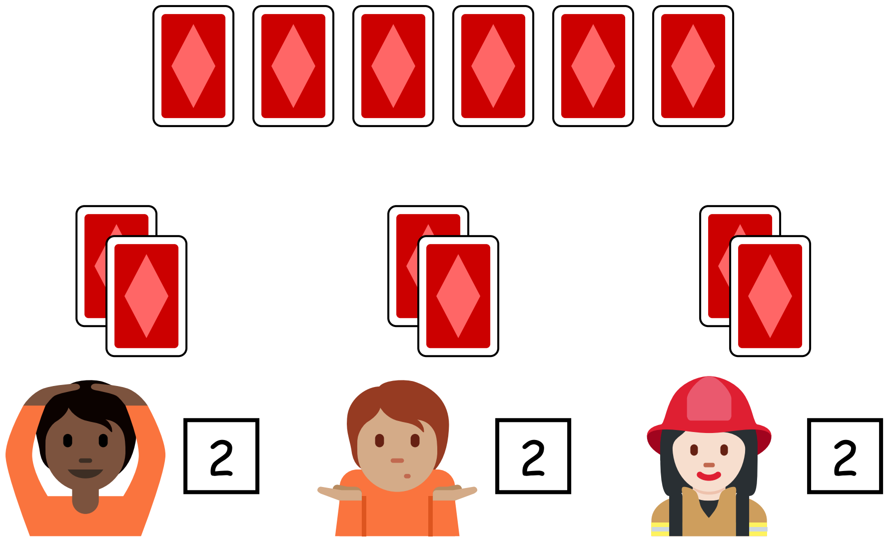




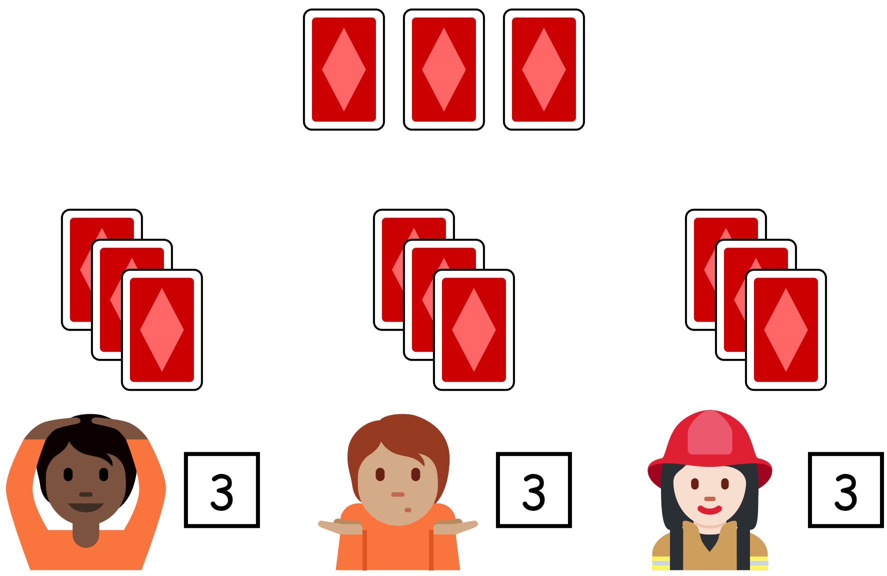
Start by giving each player one card at a time. After the first round, each player will have 1 card.



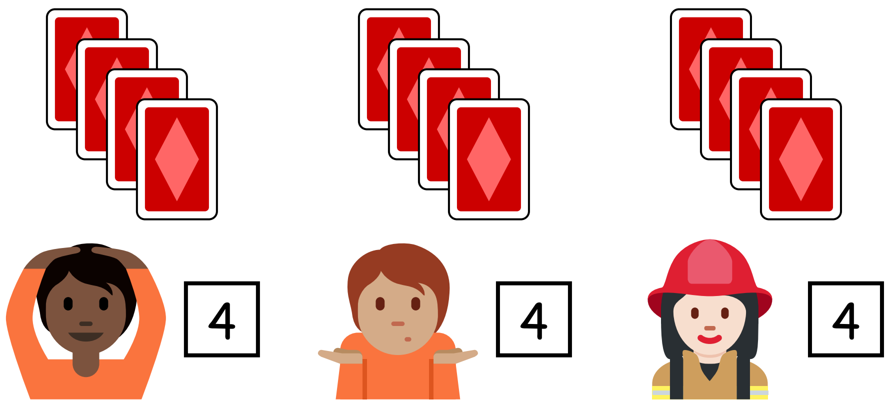
After the second round, each player will have 2 cards.



After the third round, each player will have 3 cards.



This continues until all cards are distributed.



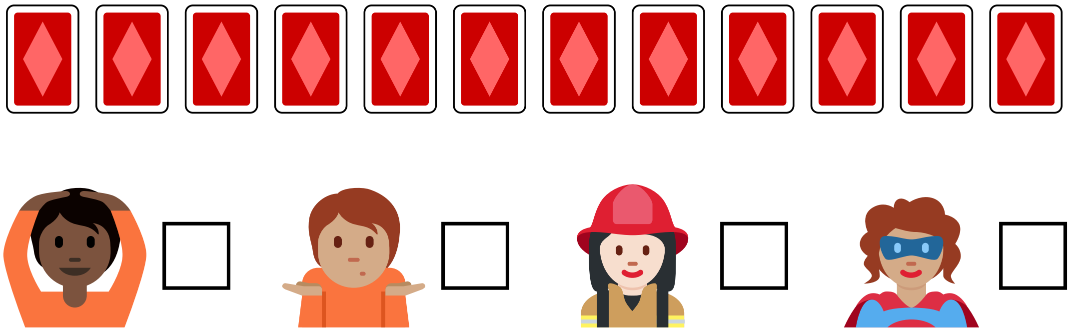
By the end, every player has 4 cards.

This process of splitting into equal parts is called *division*. In the previous example, we said that 12 cards divided among 3 players equals 4 cards per player. There are different ways to write this.

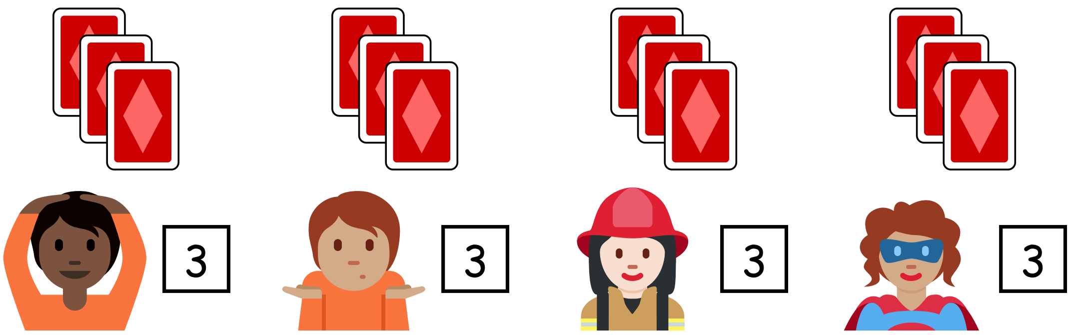
$$12 \div 3 = 12/3 = \frac{12}{3} = 4$$

Question

How many cards will each player get if you invite a new friend and play the same game with 4 players?



If we follow the same process of giving out the cards one by one, each player ends up with 3 cards.



This can be written as follows:

$$12 \div 4 = 12/4 = \frac{12}{4} = 3$$

What is division?

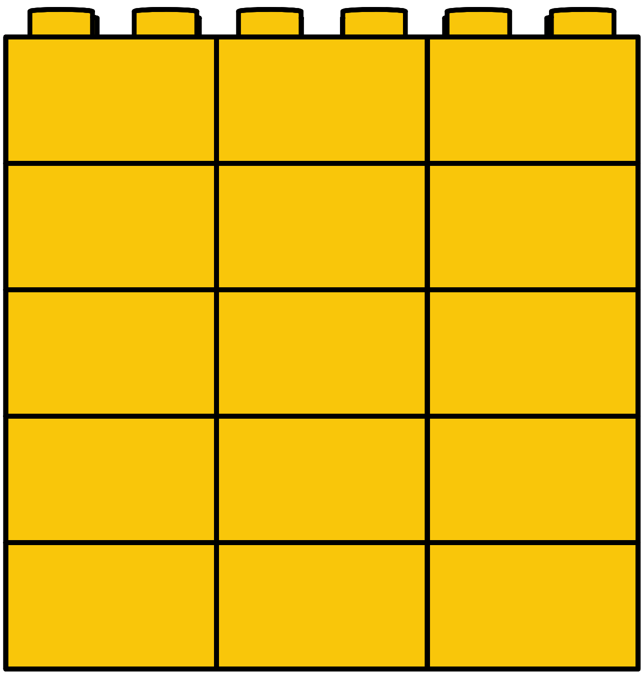
Just as subtraction is the inverse of addition, division is the inverse of multiplication.

To see this, let's review the previous results and try to connect them with multiplication. We can write the first result as follows:

$$\frac{12}{3} = 4$$

$$12 = 3 \times 4$$

For example, we determined that there are $5 \times 3 = 15$ LEGO bricks in the following image:



We can also work backwards and ask, given that there are 15 LEGO bricks, how many rows of 3 bricks can we build? $15/3 = 5$. How many columns of 5 bricks can we build? $15/5 = 3$.

In general, if an integer c is the product of two integers a and b,

$$c = a \times b$$

We known than c can be divided by both a and b:

$$c \div b = a$$
$$c \div a = b$$

This means that we can use multiplication tables to find division results.

For example, if I asked you to solve the following division problem:

$$\frac{35}{5} = \boxed{?}$$

We can look at the multiplication table to find the values where the product is 35. Then, we can check if any of the columns or rows of those values contains a 5.

×	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1		1	2	3	4	5	6	7	8	9	10
2			4	6	8	10	12	14	16	18	20
3				9	12	15	18	21	24	27	30
4					16	20	24	28	32	36	40
5						25	30	35	40	45	50
6							36	42	48	54	60
7								49	56	63	70
8									64	72	80
9										81	90
10											100

Since 35 is the product of 5 and 7, we can conclude that $35 \div 5 = 7$.

Exercises

- If I have 20 LEGO bricks, how many towers of 4 bricks each can I build?
- If my grandparents bring me and my sister 8 bombons, and we want to split them equally, how many will each of us get?
- Compute the following divisions using the multiplication table:

$$36 \div 6 = \square$$

$$48 \div 8 = \square$$

$$56 \div 7 = \square$$

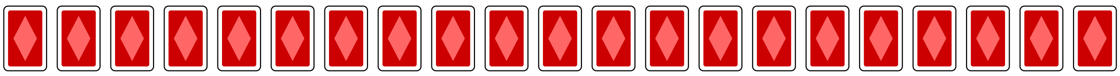
$$63 \div 9 = \square$$

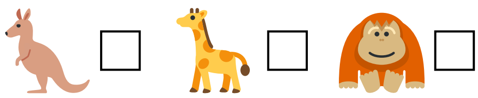
$$72 \div 8 = \square$$

$$100 \div 10 = \square$$

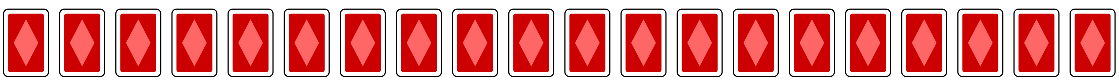
- In the following games, how many cards does each player get?

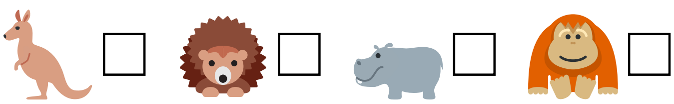
`21 cards divided by 3 players:`



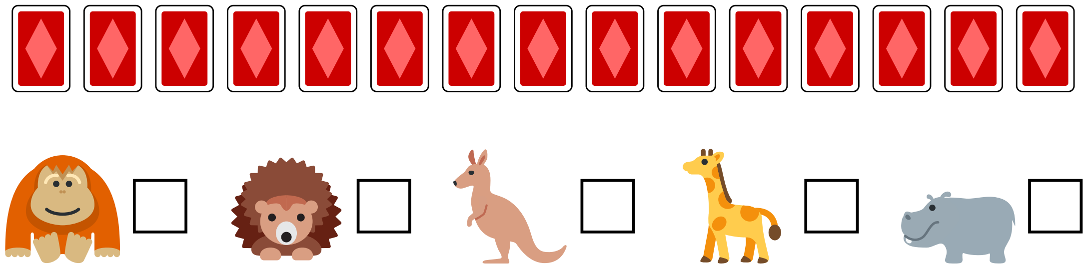


`20 cards divided by 4 players:`

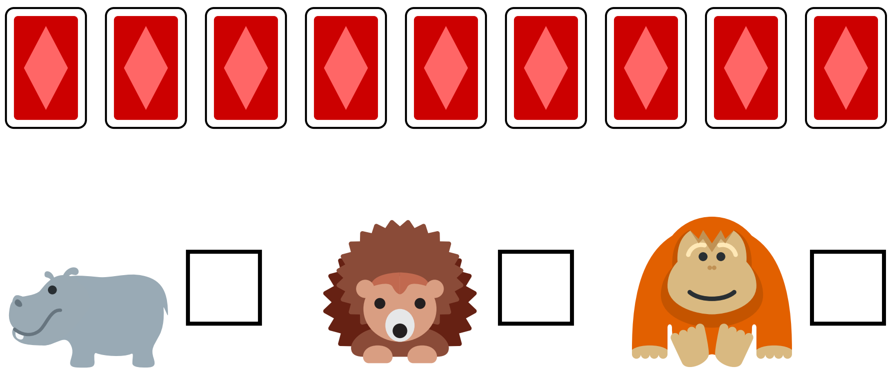




```
15 cards divided by 5 players:
```



```
9 cards divided by 3 players:
```



5.8 Multiplying and dividing by powers of 10

Multiplying and dividing by powers of 10 is a special case that can be done quickly by shifting the digits of a number. This is because 10 is the base of our decimal number system. The same idea works in any base and is how computers efficiently perform multiplication and division by powers of 2.

Given a number with multiple digits, $d_1 d_2 \ldots d_n$, and a power of 10 to multiply by, the result is simply the original number shifted to the left by the exponent of 10 (the number of zeros in the power of ten) for multiplication and to the right for division.

Remember, a power of 10 is a 1 followed by 0s.

$$\underbrace{10 \times 10 \times \cdots \times 10}_{n \text{ times}} = 10^{n} = 1\overbrace{000 \cdots 000}^{\text{Number of zeros after the 1}}$$

This is what happens when we multiply a general number $d_1 d_2 \dots d_n$ by a power of 10:

$$d_1 d_2 \dots d_n \times 10^n = d_1 d_2 \dots d_n \times 1\underbrace{0 \cdots 0}_{n \text{ zeros}} = d_1 d_2 \dots d_n \underbrace{0 \cdots 0}_{n \text{ zeros}}$$

The following examples should make this concept clearer.

$$5 \times 10^1 = 5 \times 10 = 50$$
$$5 \times 10^2 = 5 \times 100 = 500$$
$$753 \times 10^4 = 753 \times 10000 = 7530000$$

Exercises

Perform the following computations:

$$2 \times 10 = \boxed{}$$
$$20 \times 10 = \boxed{}$$
$$123 \times 1000 = \boxed{}$$
$$123 \times 10^3 = \boxed{}$$

$$57 \times 10^2 = \boxed{}$$

$$21 \times 100 = \boxed{}$$

$$1000 \div 10 = \boxed{}$$

$$1200 \div 100 = \boxed{}$$

$$5700 \div 10 = \boxed{}$$

$$5700 \div 10^2 = \boxed{}$$

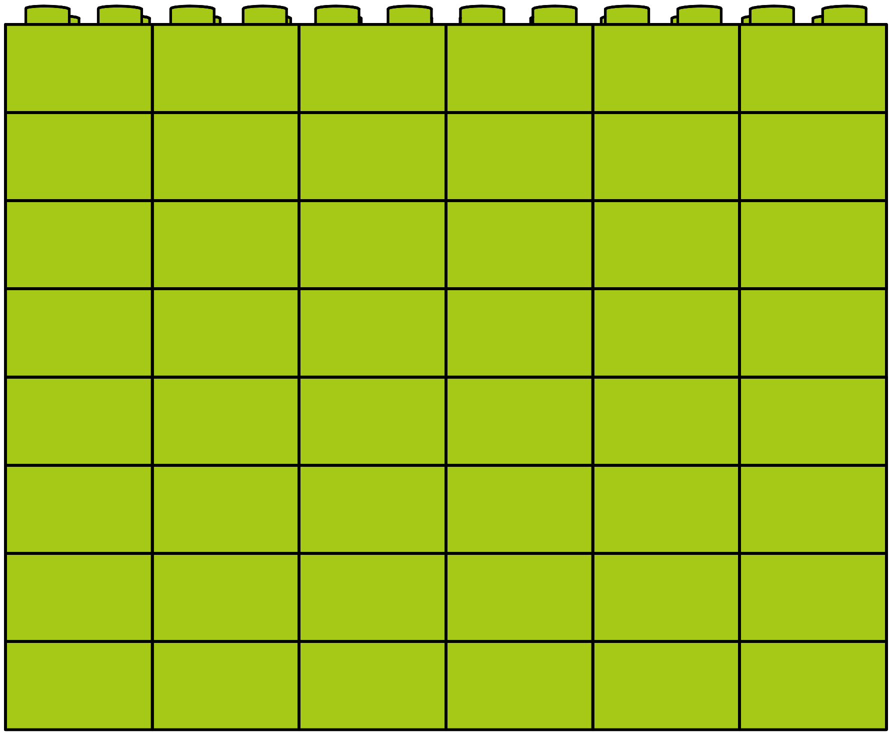
6 Multiplication and geometry

We began our journey of learning multiplication by discerning patterns. Then, we learned how to express those patterns as the product of two or three integers. Finally, we learned how to compute those products using multiplication tables. Now, we are cycling back and formalizing the idea of patterns, making a link between multiplication and geometry. Specifically, we are exploring how multiplying two natural numbers produces a two-dimensional object called a plane and how multiplying three natural numbers produces a three-dimensional object called a volume.

Remember that in Section 2.6, we defined a dimension as one of the directions in which you can move. Here, N represents the number of dimensions.

N	Object	Symbol	How can we move?	How to construct?
0	Point	•	We cannot move.	
1	Line		We can move only forward and backward.	d
2	Plane		We can move forward and backward, and also left and right.	$d \times d = d^2$
3	Volume		We can move left, right, forward, backward, and jump up and down.	$d \times d \times d = d^3$

An area is a measurement that indicates the size of a two-dimensional object. For instance, the area of the wall pictured below equals the number of LEGO bricks that comprise it.



The area is equal to the total number of green bricks: $6 \times 8 = 48$.

Count the bricks if you want to be convinced.

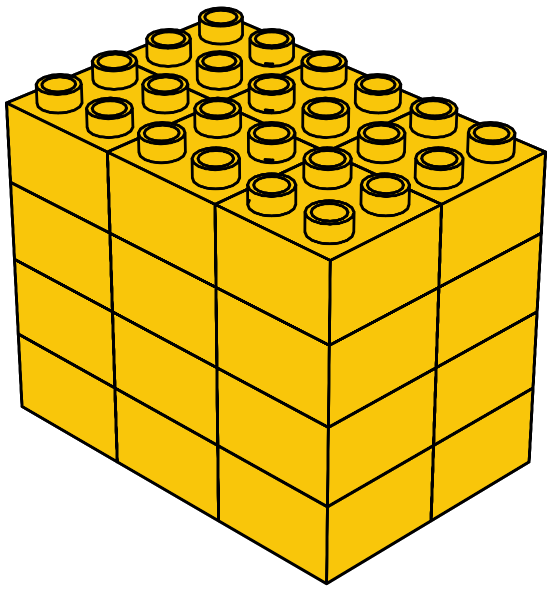
What is an area?

The area of a wall that is w LEGO bricks wide and h LEGO bricks high is equal to the number of bricks in the wall. We can compute it as follows:

$$\text{area} = w \times h$$

Note that this is true for walls of any width, w, and height, h.

Similarly, volume is a measure of the size of a three-dimensional object. For example, the volume of the cube below is equal to the number of LEGO bricks that make up the cube.



The volume is equal to the total number of yellow bricks: $3 \times 4 \times 2 = 24$.

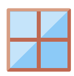
What is an volume?

The volume of a box with width w, height h, and depth d is the number of LEGO bricks that can fit inside the box. We can compute it as follows:

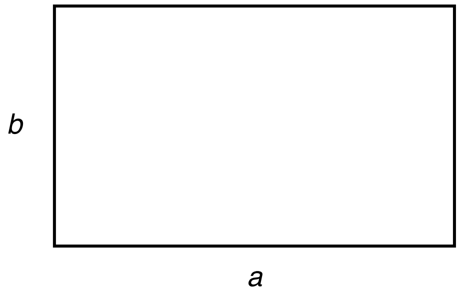
$$\text{volume} = w \times h \times d$$

Keep in mind that this is true for any box with a width of w, a height of h, and a depth of d.

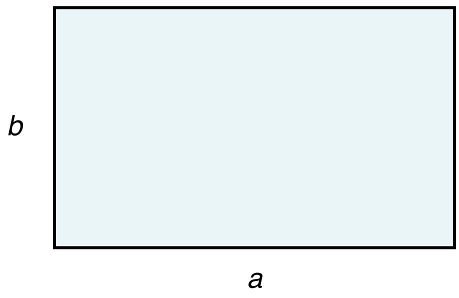
6.1 Calculating rectangular areas

Initially, we will focus on rectangular objects and then move on to other shapes. Examples of rectangular shapes include doors  , windows  and flags.

Given a general rectangle with width a and height b:



The area of the rectangle is the space inside it.

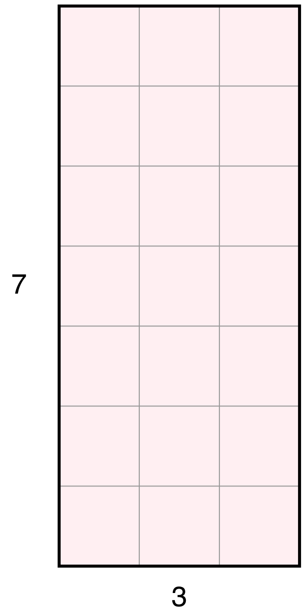


And is calculated as:

$$\text{Area} = a \times b$$

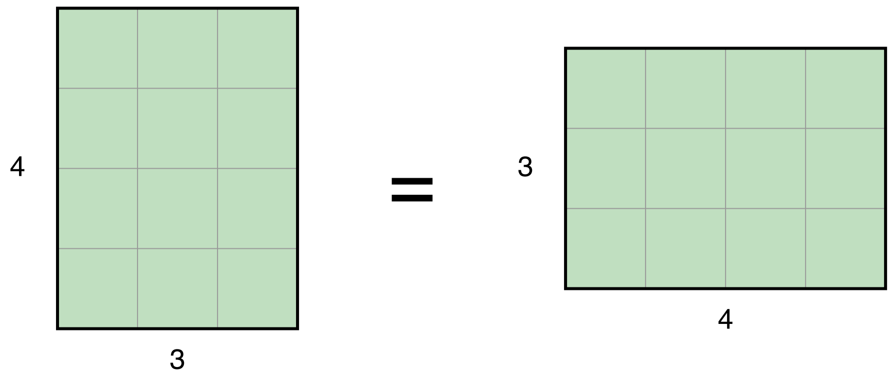
For example, the area of the rectangle below is 21, which is the number of unit squares inside it. Count them and see for yourself!

$$\text{Area} = 3 \times 7 = 21$$



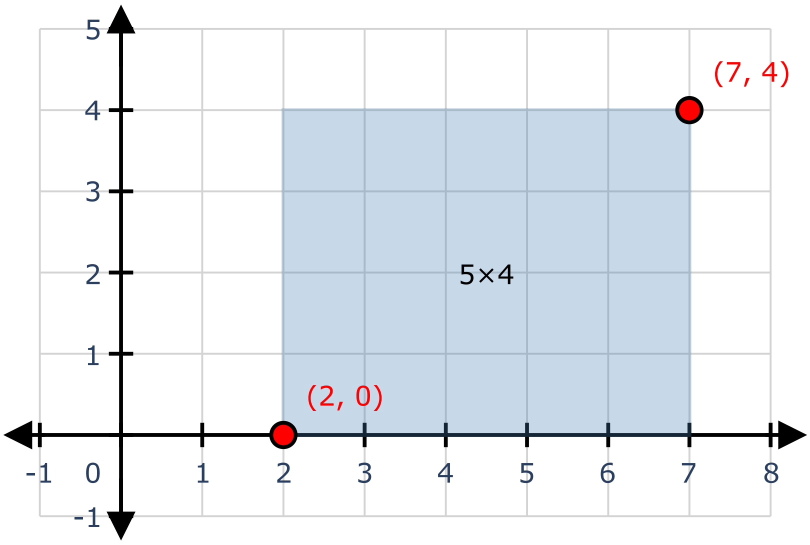
These rectangles have the same area, which is equal to:

$$\text{Area} = 3 \times 4 = 4 \times 3 = 12$$



Drawing areas on the Cartesian plane

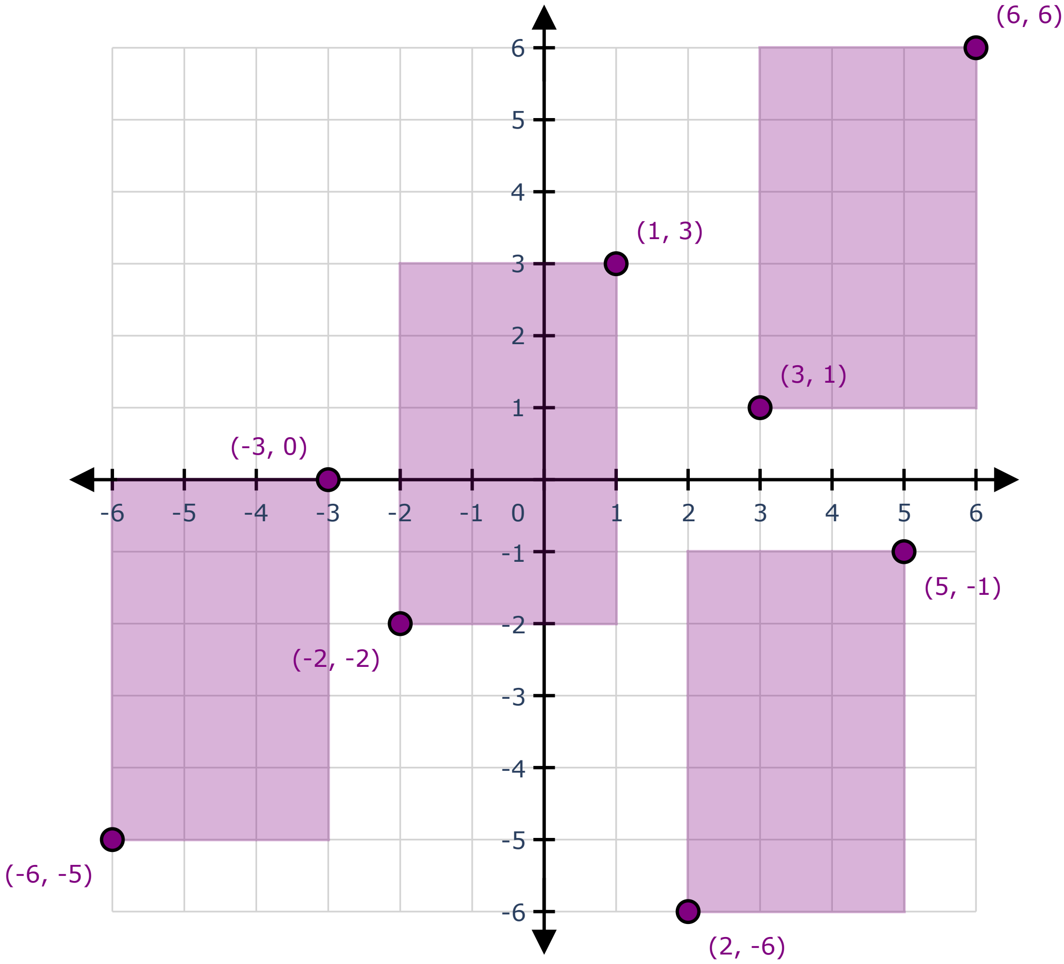
We can also draw rectangles on the Cartesian plane.



In this case, the area is equal to the product of the horizontal and vertical distances between $(7, 4)$ and $(2, 0)$.

$$\text{Area} = (7 - 2) \times (4 - 0) = 5 \times 4 = 20$$

The area depends only on the distances, which are always positive, and not on the location of the rectangle. For example, the area of all the rectangles below is 15, regardless of their location, even if they are in the negative quadrants.

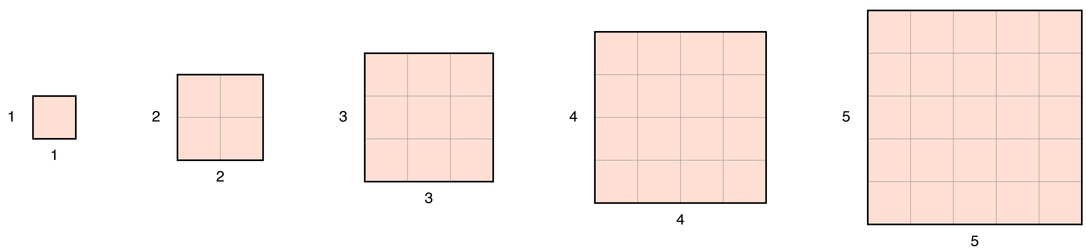


Squaring a number

We learned that when we multiply a number by itself, we say that we are "squaring" it. These values appear diagonally in the multiplication tables.

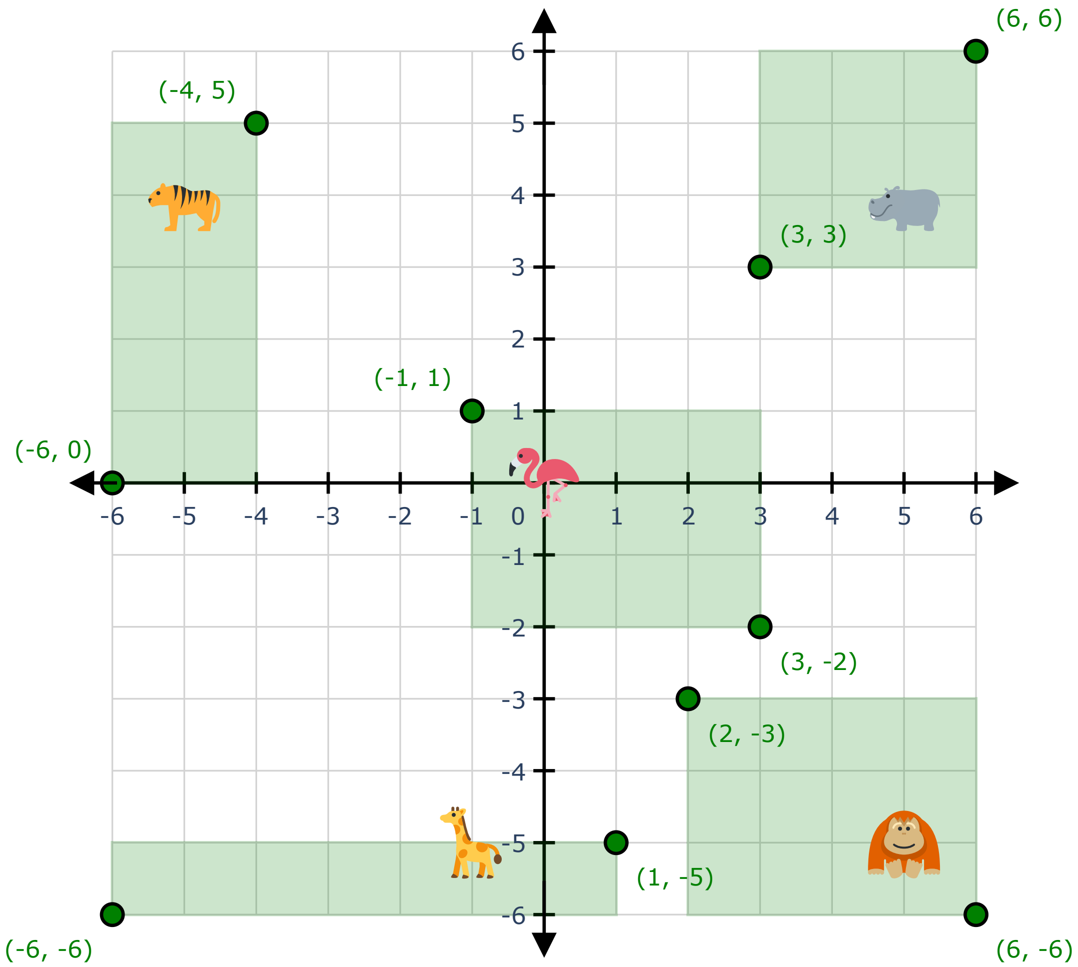
×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15
4	0	4	8	12	16	20
5	0	5	10	15	20	25

Now, we can see why it's called "squaring." The resulting rectangle is a square, with all sides of the same length. The rectangles below represent the same values as the highlighted elements on the diagonal of the multiplication table above. Count the number of unit squares and compare.



Exercises

Find the total area of the zoo. The total area is the sum of all the green spaces.

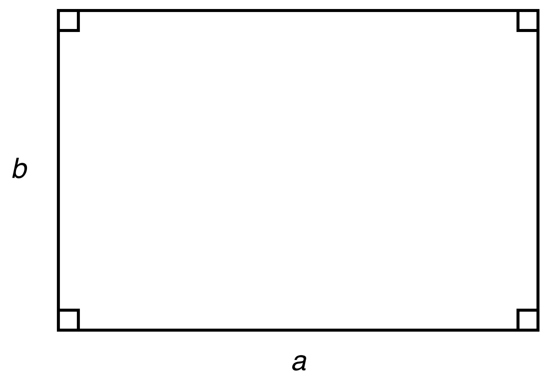


6.2 Triangles 📐

Triangles are the simplest type of polygon, with the fewest sides. They are so important in mathematics that an entire branch of the subject, trigonometry, is dedicated to studying them.

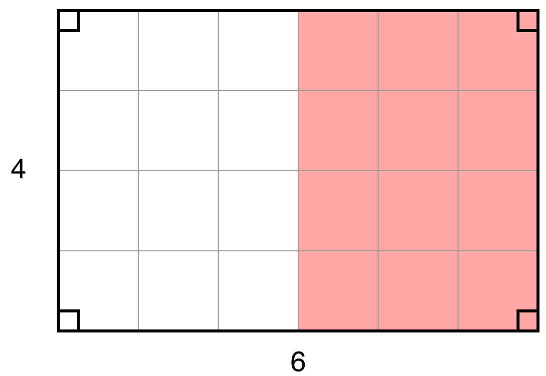
We can apply our knowledge of how to calculate the area of a rectangle to calculate the area of a triangle. Let's see how.

The corners of a rectangle are right angles. This means that the two lines meeting at each corner are perpendicular to each other $\perp$. Examples of right angles can be seen where two walls meet or where a wall meets the floor or ceiling. We can accentuate this with the following symbols:

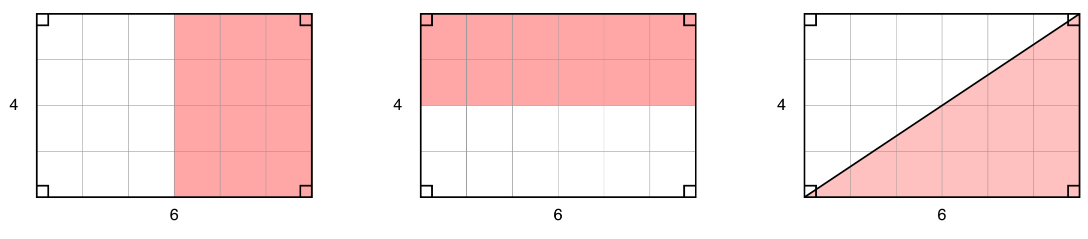


If we split a rectangle in half, the resulting areas are each half of the area of the original rectangle, right?

For example, the area of the rectangle below is $6 \times 4 = 24$, and the area of the half red rectangle is $\frac{6\times4}{2} = \frac{24}{2} = 12$. Count and convince yourself.



This is true no matter how we split the rectangle in half.

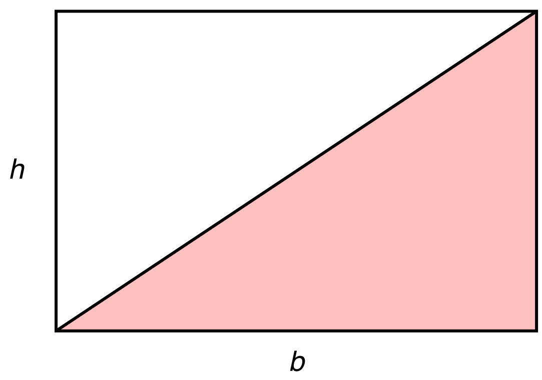


Question

Why is this fact useful?

Because we can use it to calculate the area of a triangle. A triangle is half of a rectangle, so its area is half that of the rectangle.

The resulting triangles are called right triangles because they have a right angle.

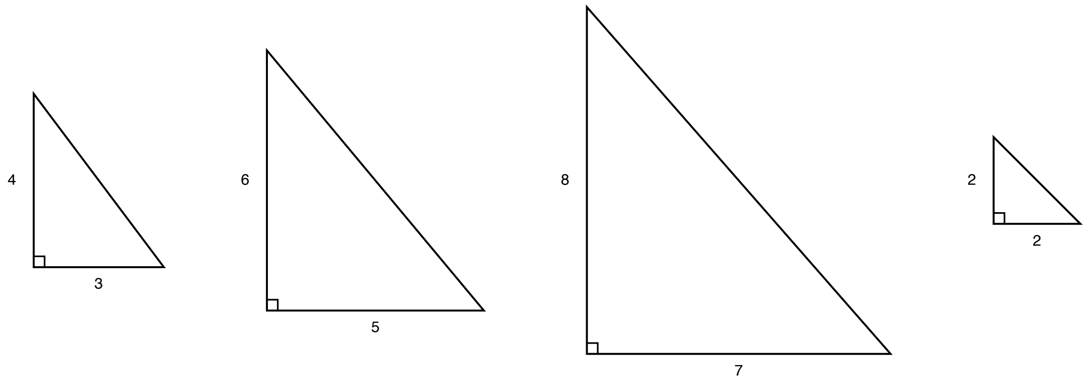


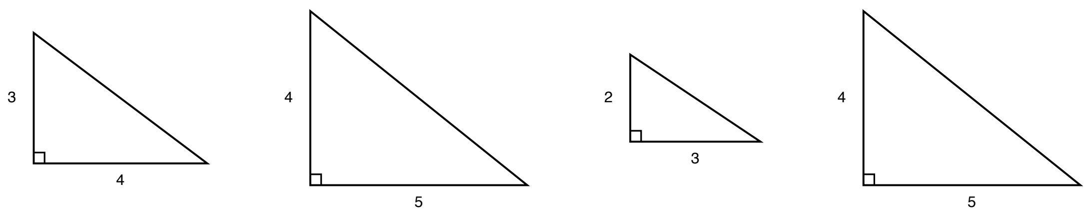
In general, for a right triangle with a base of b and a height of h, the area is given by the formula:

$$\text{Area} = \frac{b \times h}{2}$$

Exercises

Calculate the area of the following triangles:



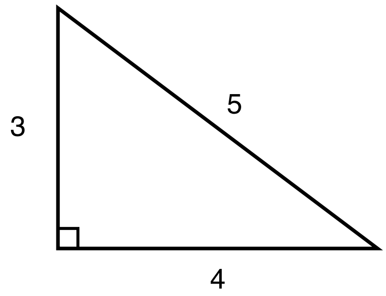


6.3 Proving the most famous theorem with paper 📜 and scissors ✂️

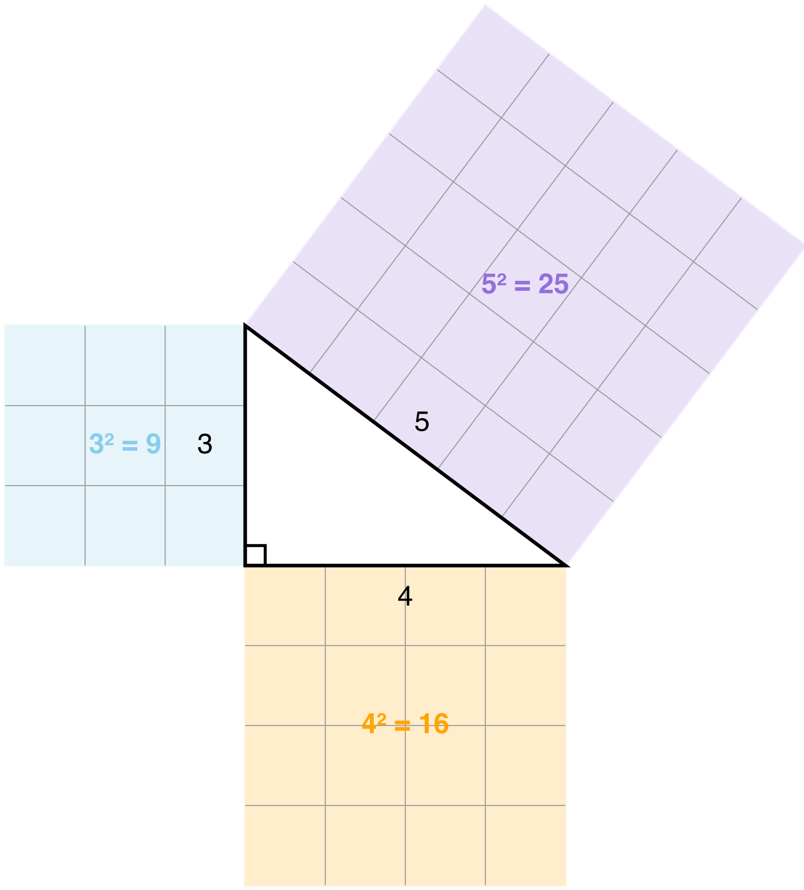
Right triangles are important and useful. They have been studied since ancient times. One reason they are important is the relationship between the lengths of their sides. For example, you can use this relationship to calculate the shortest distance between two points on a plane.

I am going to show you the geometric relationship between the sides of a right triangle.

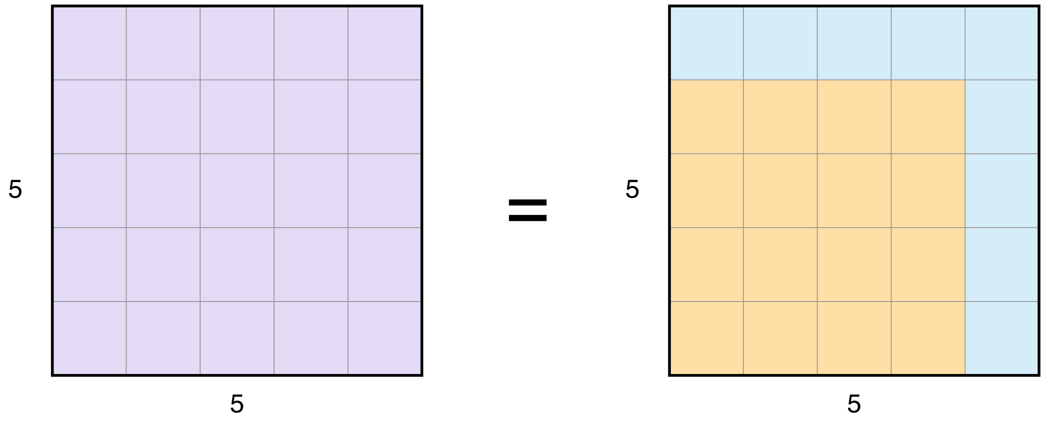
First, we start with a right triangle.



Now we are going to square the sides:



Observe the areas that they define:

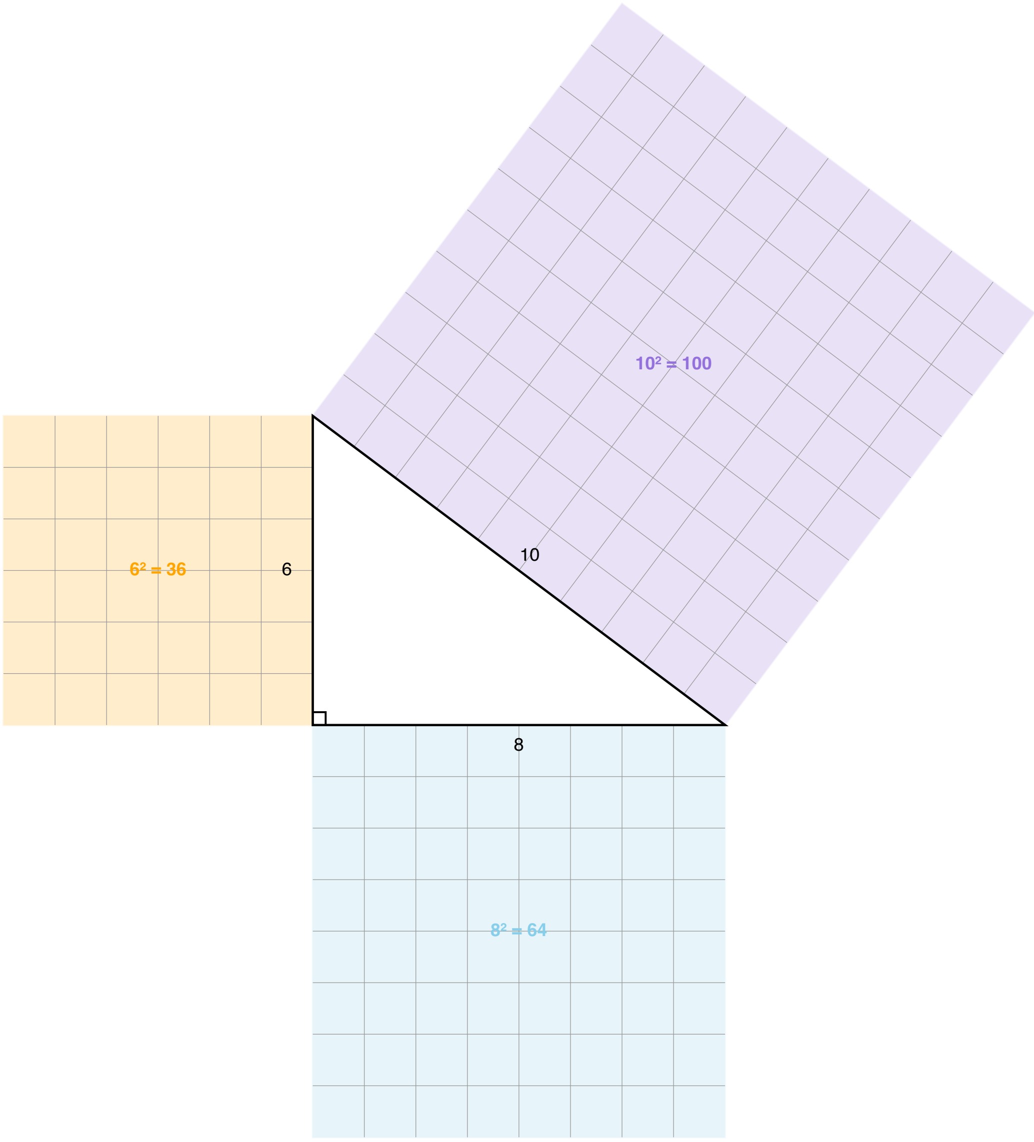


> **Question**
>
> What relationship do you see between the areas of the squares on the sides of the triangle?

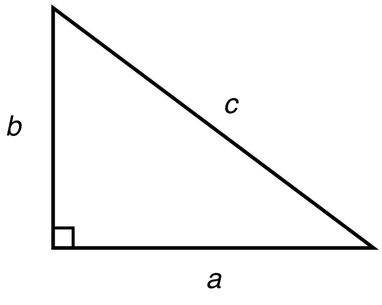
That's right. For the triangle above, we have:

$$3^2 + 4^2 = 5^2$$

Here is another example:



Does this hold true for any right triangle? It does; that result is called the Pythagorean Theorem.



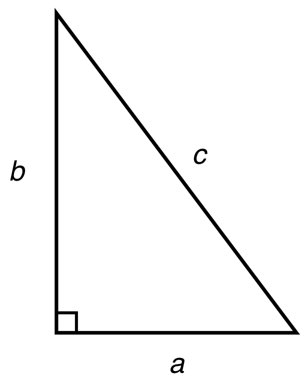
$$a^2 + b^2 = c^2$$

What is a proof?

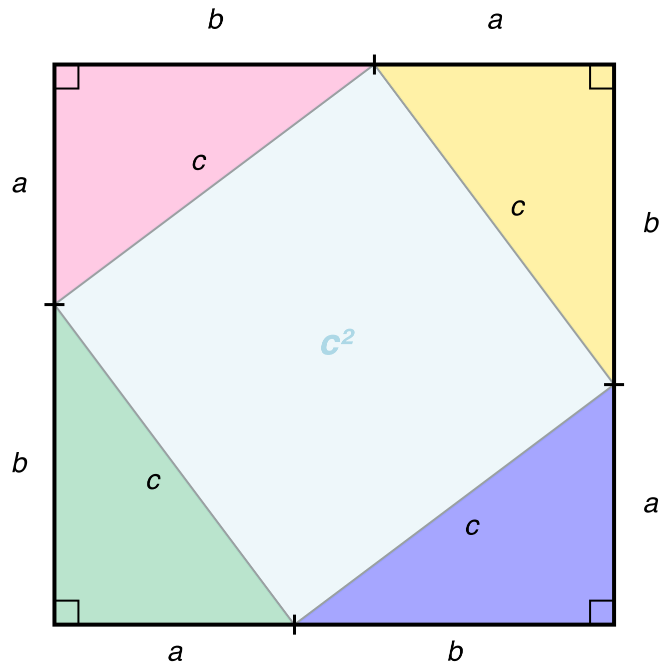
How can we be sure that this relationship holds for any right triangle? In mathematics, the only way to know for sure is to prove it. A proof is a logical argument that demonstrates the truth of a statement.

This will be our first proof.

Let's start with a right triangle whose sides have lengths a, b, and c.



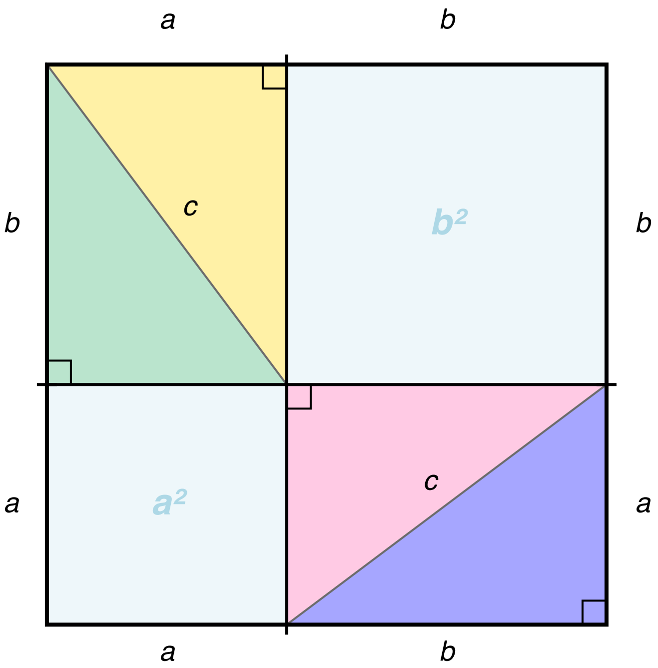
Now, we are going to arrange the four copies of the triangle inside a square with sides of length $a + b$. We will color the triangles different colors, but keep in mind that they all have the same dimensions.



Note that rearranging the triangles in this way creates a square with side length c and area c^2. This area is also equal to the area of the large square minus the area of the four triangles. Right?

Now, we are moving the triangles inside like this:

- The purple triangle remains where it is.
- We move the pink triangle so that its c side touches the c side of the purple triangle.
- Then, we move the green triangle to the top of the big square.
- Finally, we move the yellow triangle to the left of the big square.



The light area has not changed; it is still the area of the large square minus the areas of the four triangles. By rearranging the triangles, we demonstrated that:

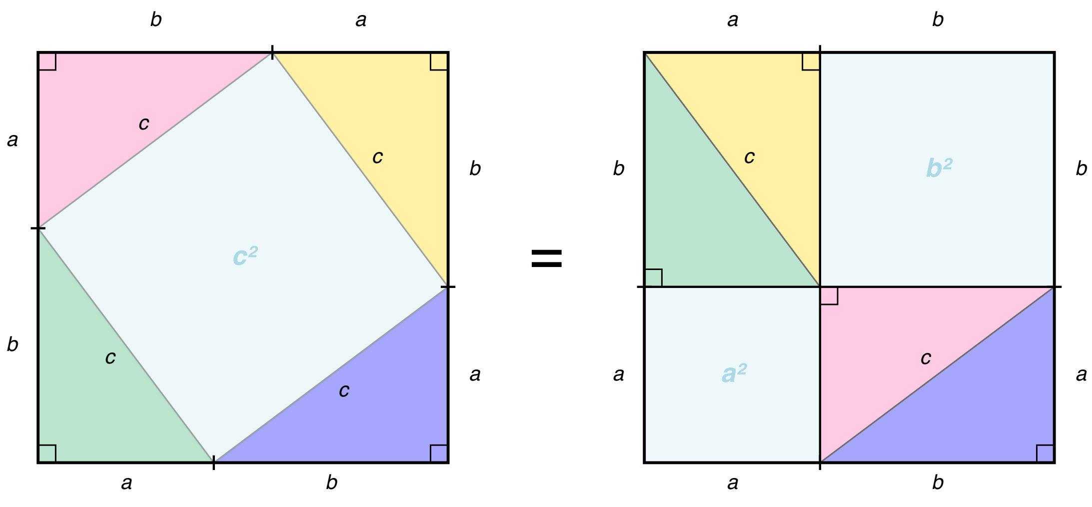


Figure 6.1: All the areas are the same in both figures.

$$c^2 = a^2 + b^2$$

The Pythagorean theorem holds for any right triangle ■. Congratulations! That's your first proof! 🥳.

We began this book by alluding to the proof of the most elusive theorem in the history of mathematics. That theorem is called Fermat's Last Theorem, and it is closely related to the Pythagorean Theorem. The Pythagorean Theorem states that, for any right triangle:

$$a^2 + b^2 = c^2$$

However, Fermat's theorem states that this relationship only holds true for squares and not for cubes or higher powers.

$$a^n + b^n = c^n \quad \text{only for } n = 0, 1, 2$$

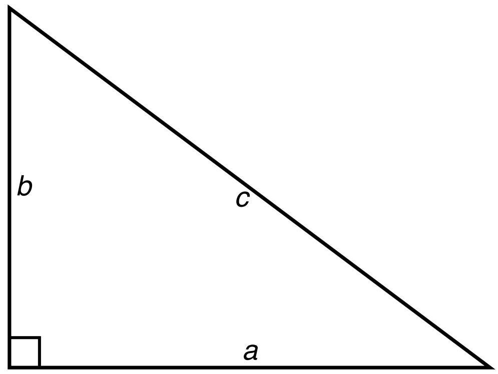
But proving that one is a bit more difficult, so we are going to leave it for later!

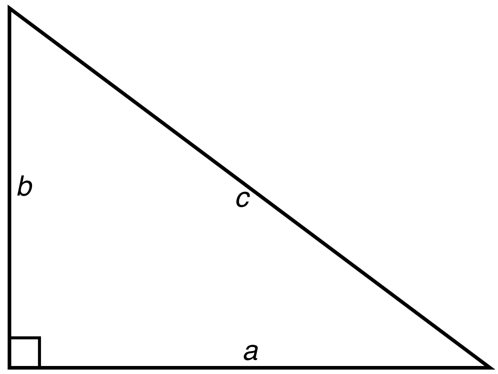
😉😉

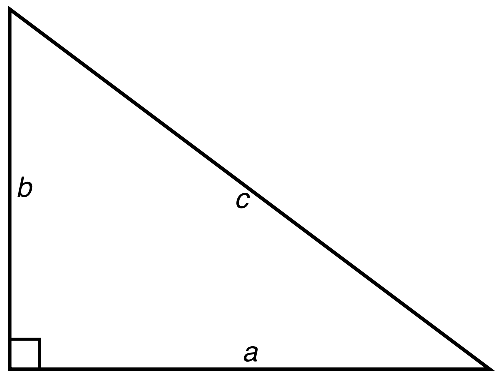
Exercises

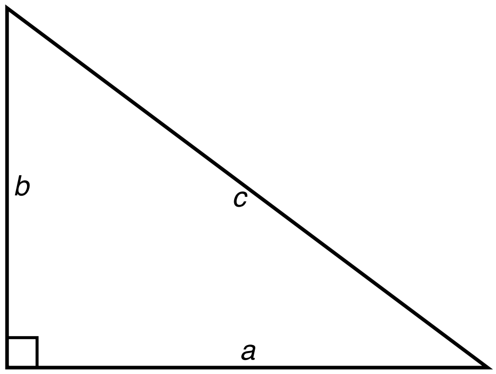
We are going to do some crafts. Have paper or cardboard and scissors ready.

Draw or print four copies of the following triangle and color them in different colors.

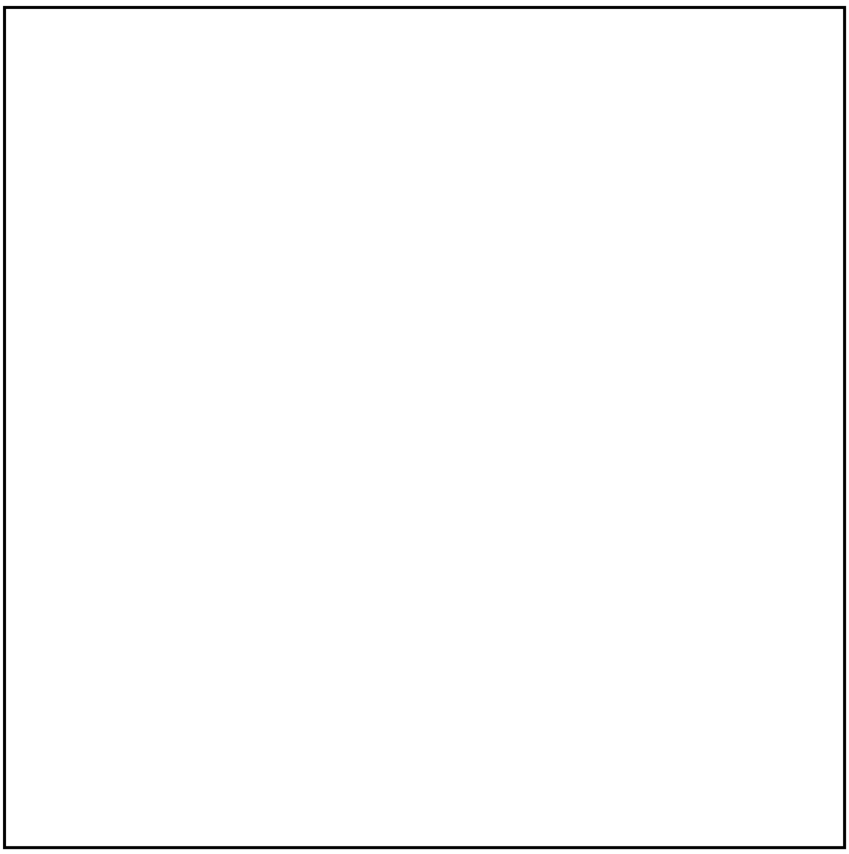

b
c
a


b
c
a


b
c
a



Draw or print the following square:

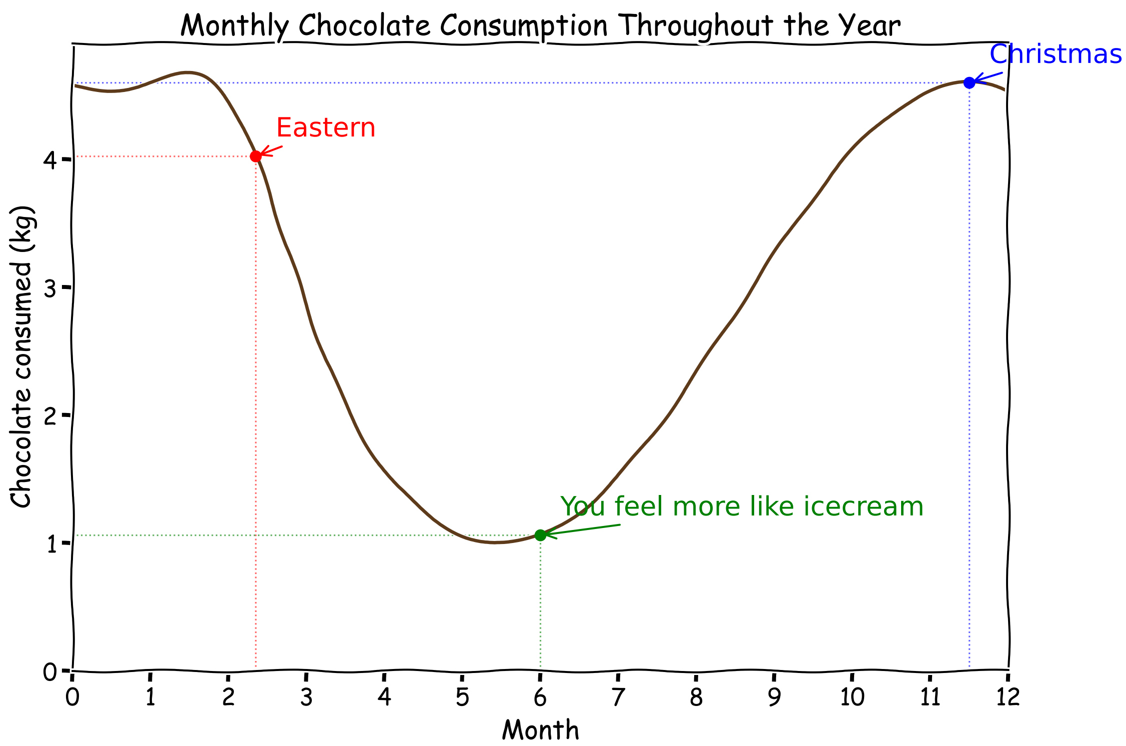


Use the colored triangles inside the square to illustrate the Pythagorean Theorem.

6.4 How much chocolate do you eat per year? 🍫🍫🍫

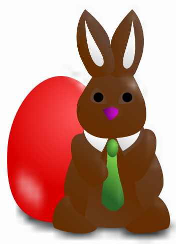
I bet you love chocolate! You're a chocolate monster!

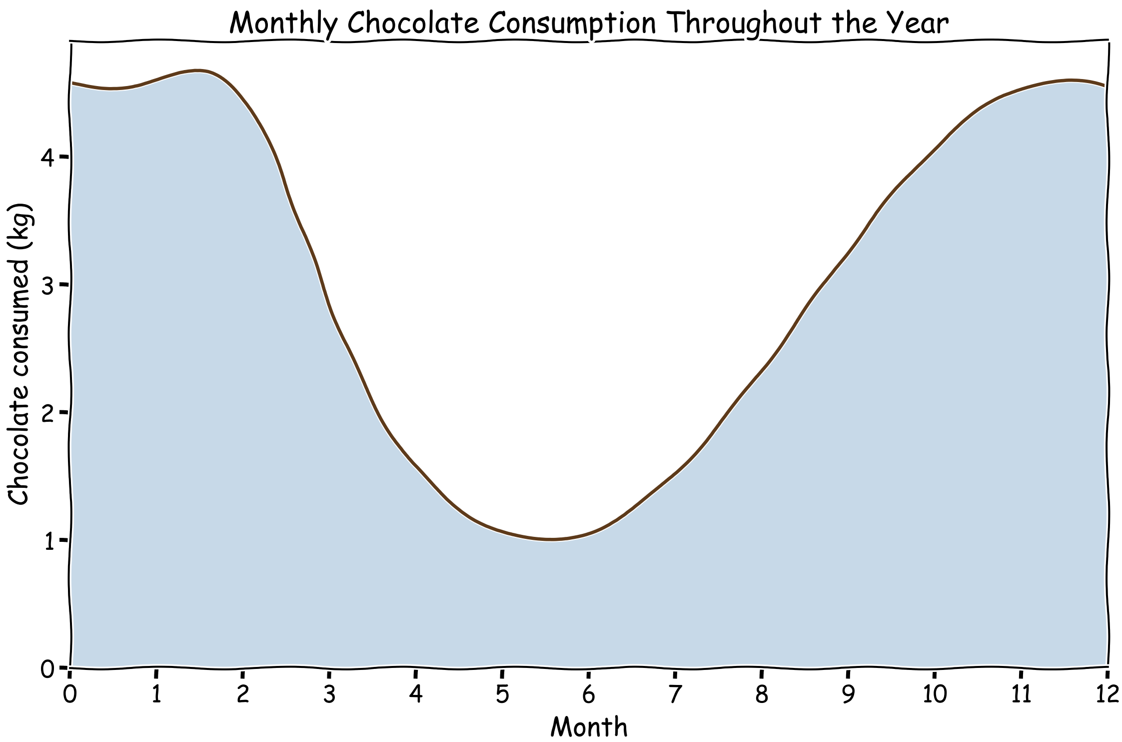
The following graph shows how much chocolate you eat each year. The x-axis represents the 12 months of the year. For each month, you can see how many kilograms of chocolate you eat. For example, in June (month 6), you only eat about 1 kg. This is probably because you prefer gelato when it's hot outside. On the other hand, you eat a lot of chocolate around Easter and Christmas—around 4 kg or more.



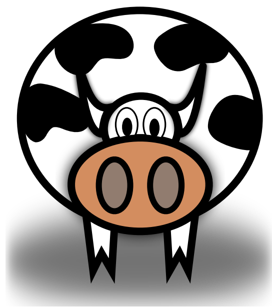
How much chocolate do you eat per year?

In order to answer this question, you must calculate the total area under the above curve.



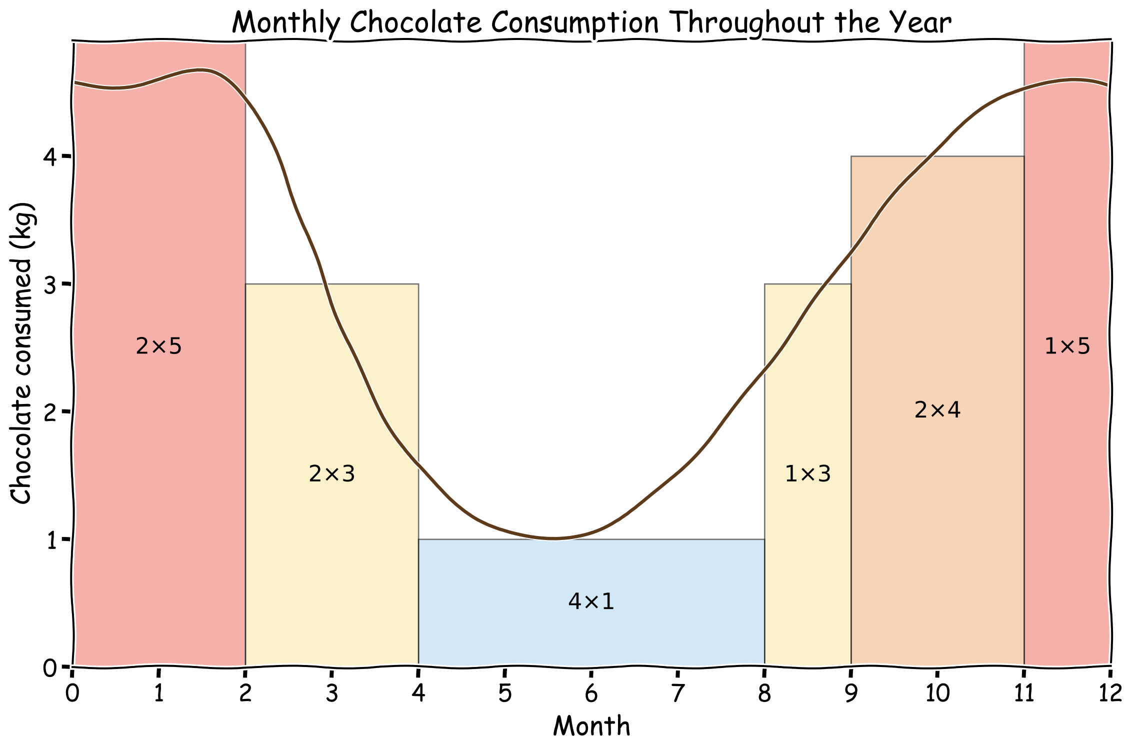


But wait a minute. We don't know how to do that yet. We don't know how to do it directly yet, but we can use our knowledge of how to calculate the area of rectangles to *approximate* it.

> **Spherical cows**
>
> Approximating means cutting corners and simplifying. For example, under some circumstances, if a quantity is 9, you can approximate it as 10 because 10 is easier to work with. Engineering and physics make heavy use of this technique. Sometimes it is useful to assume that things are points, that something is constant, and even that cows are spheres.
>
> 

We can approximate the area under the chocolate consumption curve using rectangles. The total value would be the sum of the areas of the rectangles. This is not an exact value, but it will give you an estimate.

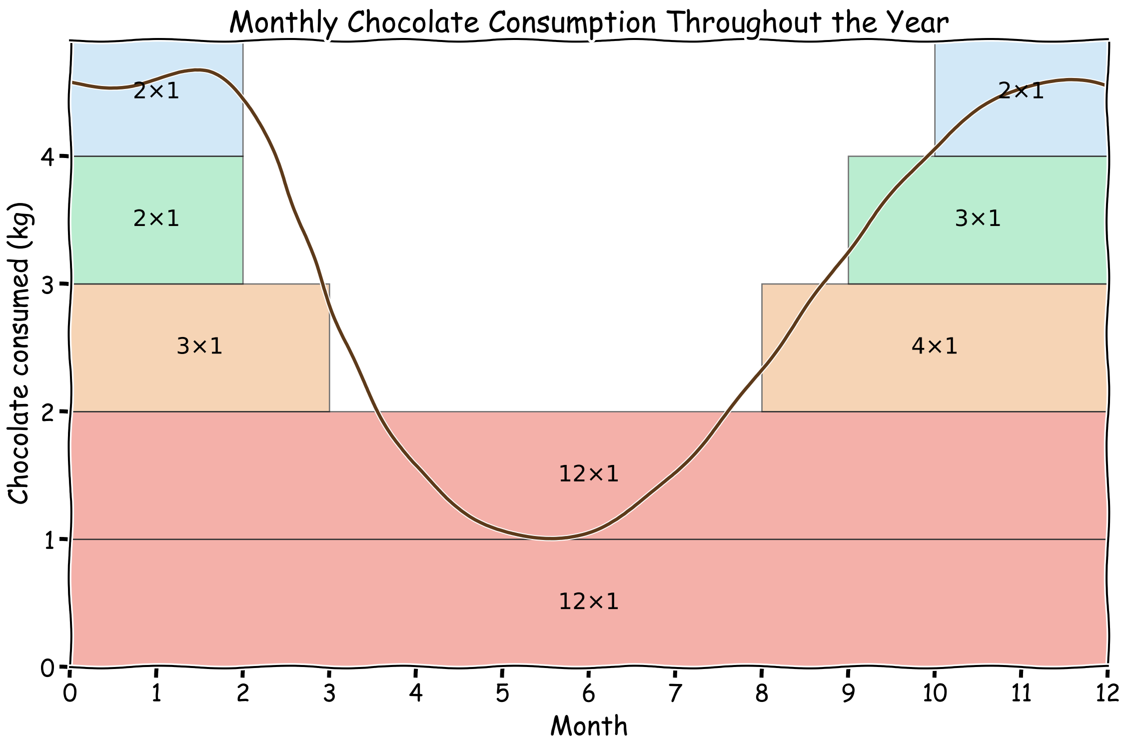
Compute your total annual chocolate consumption by adding up the areas of all the rectangles.



Exercises

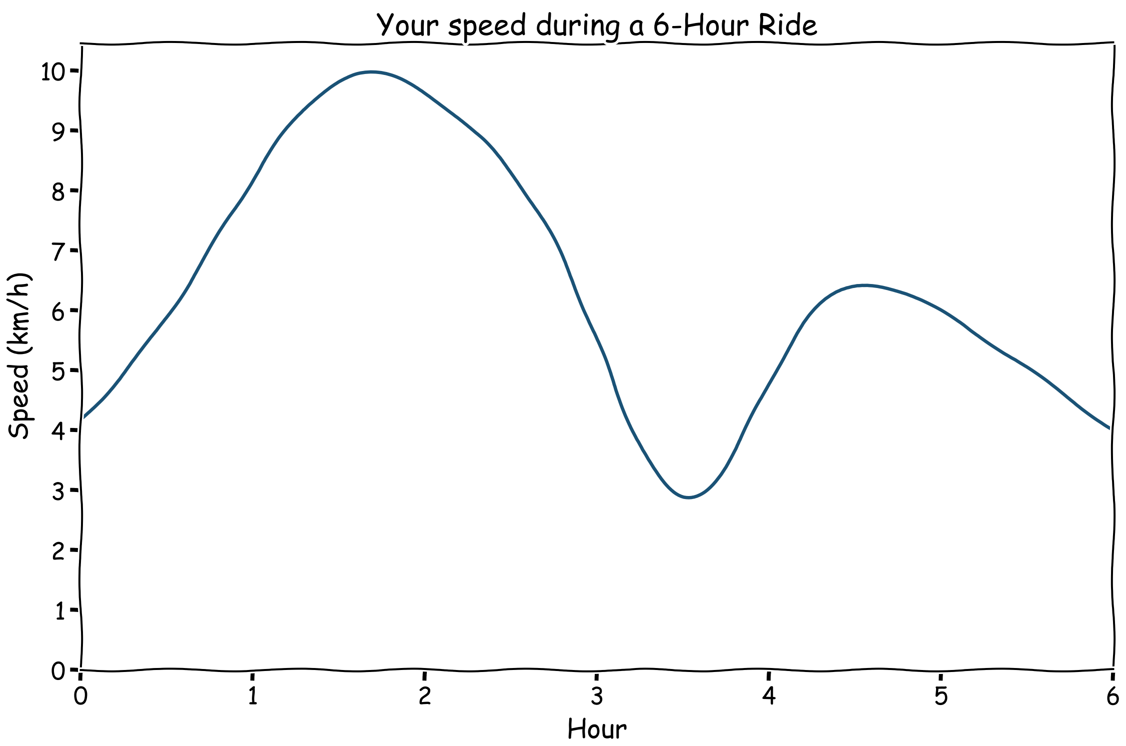
- Based on the previous graph, compute the total amount of chocolate.

There is another way to calculate the area under the curve. Instead of using vertical rectangles, use horizontal rectangles. Compare the total area of the horizontal rectangles to the previous one. They should be similar, but not exactly the same.

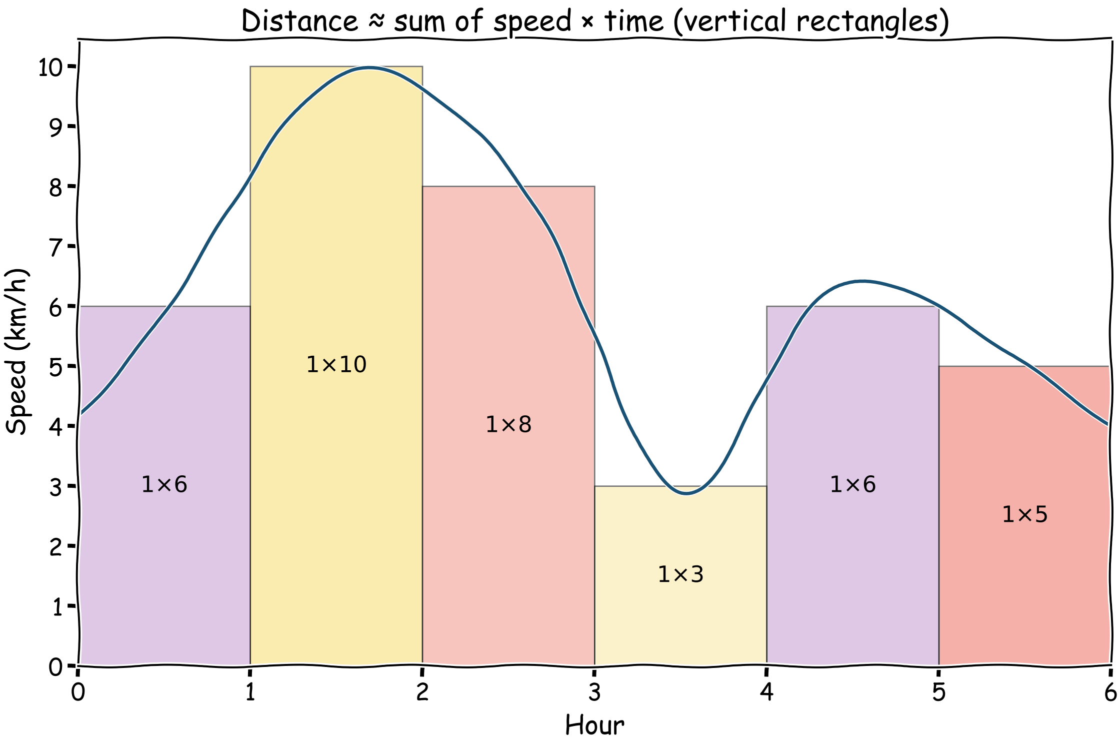


Imagine going on a bike tour with your family. Your family loves cycling 🚴‍♀️🚴🚴‍♂️ as much as you love chocolate.

The following graph shows your speed during the tour.



The area under the curve represents the total distance traveled. How far did you travel?

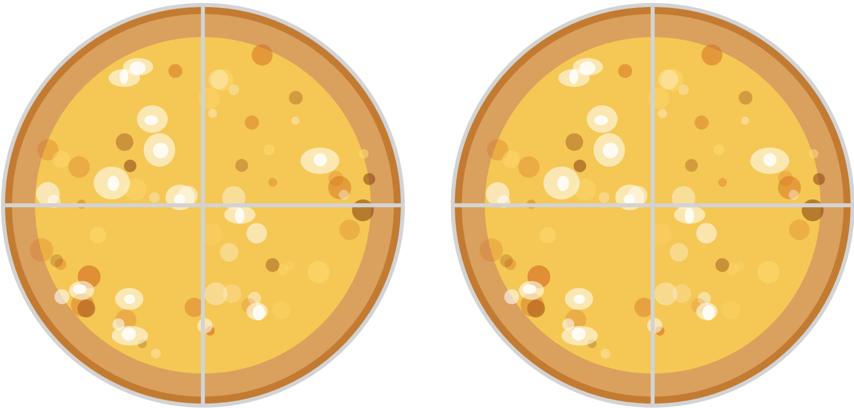


7 Pizza Math 🍕

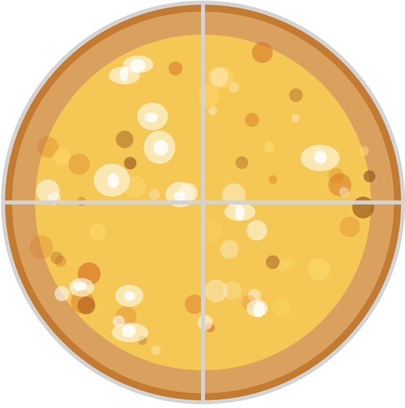
7.1 How much pizza do you eat?

I know for a fact that, besides being a chocolate 🍫 and a gelato 🍨 monster, you also love pizza 🍕. Margherita, of course.

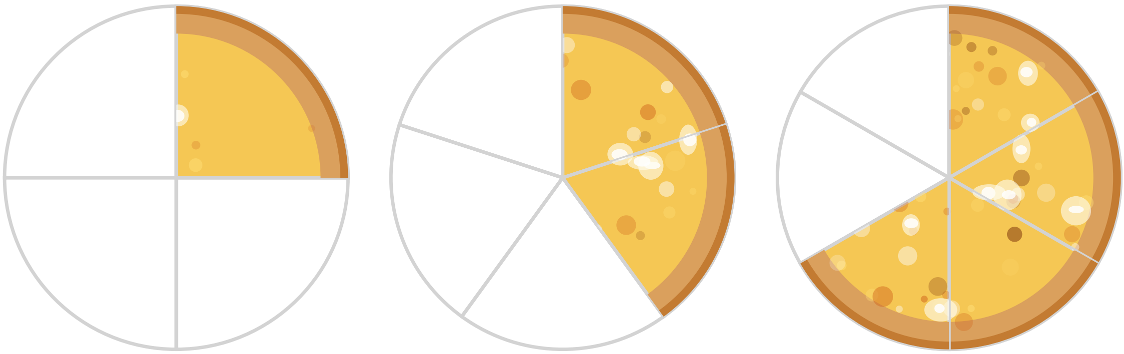
Some times you eat 2 pizzas in a single day.



Other times, you eat just 1 pizza.



But some times you are not that hungry and you just eat a *fraction* of the pizza.



> **Question**
>
> How should these quantities be represented?

So far we have operated in the realm of the integers, which represent *whole* quantities. The very word *integer* means *whole* or *complete*.

In this chapter, we will learn about another set of numbers, called the *rational* numbers $\mathbb{Q}$, which can be used to represent *parts* or *fractions* of whole quantities. We will learn that just as the natural numbers $\mathbb{N}$ are included in the integers $\mathbb{Z}$, the integers are included in the rational numbers:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$$

7.2 How do you name these new numbers?

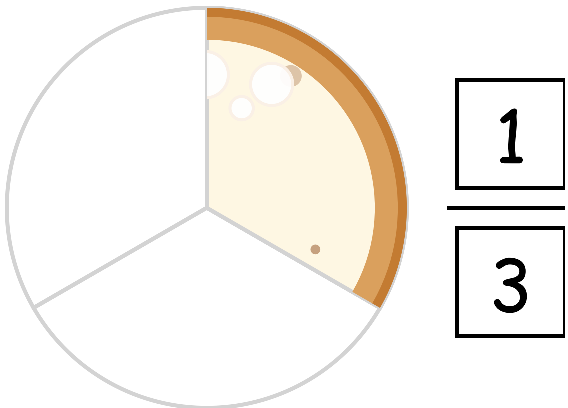
Naming fractions is easy. First, you need the number of parts into which the whole unit is divided. That number is called the *denominator*. Then, you need to know how many of those parts are included; that number is called the *numerator*.

How many slices you eat

$$\frac{\text{numerator}}{\text{denominator}}$$

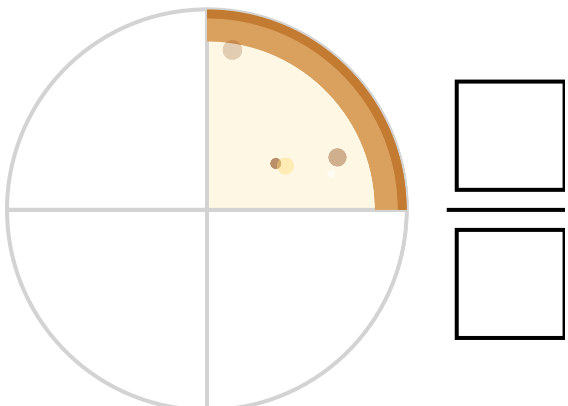
Number of slices the pizza is divided into

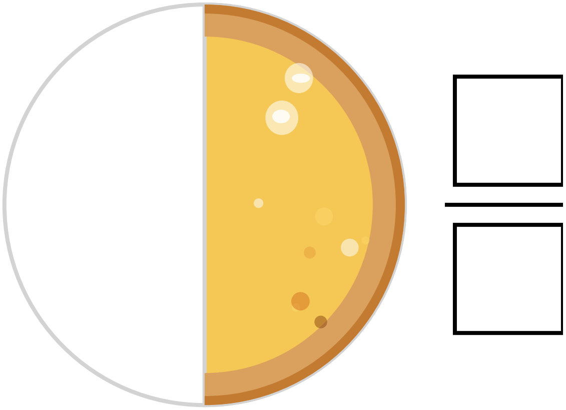
Now, let's see how to represent these numbers. For example, the following quantity represents the fraction $\frac{1}{3}$, which is read as "one third." This means that the whole pizza is divided into 3 parts, and you eat 1 of those parts.

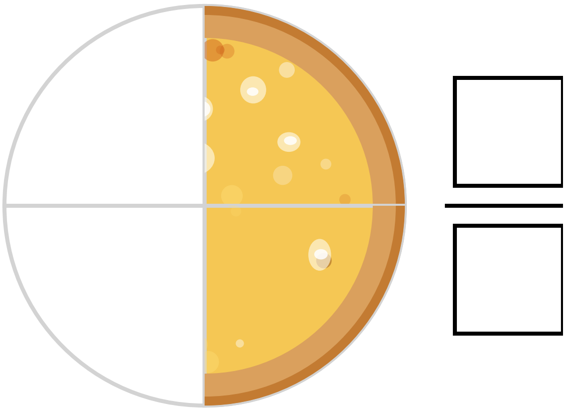


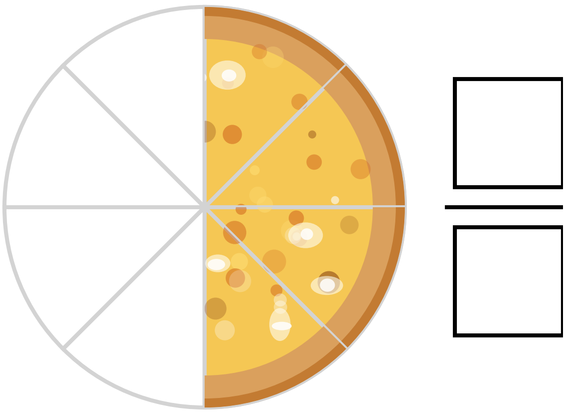
Exercises

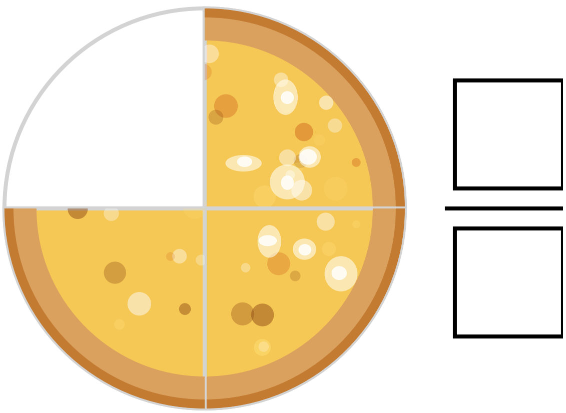
Write down the following fractions:

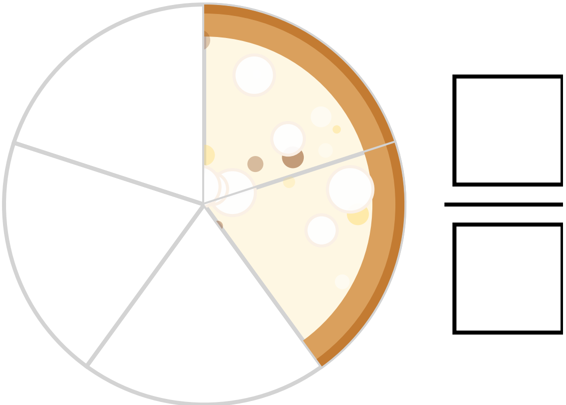


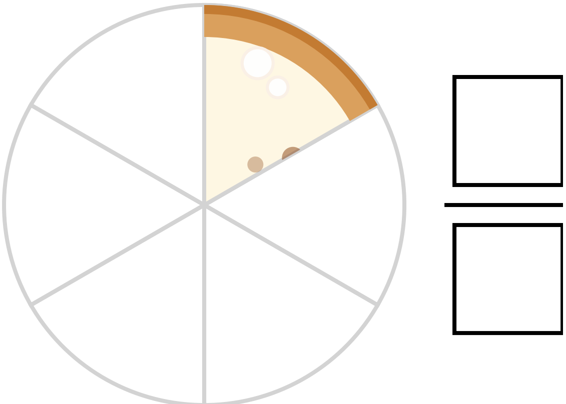


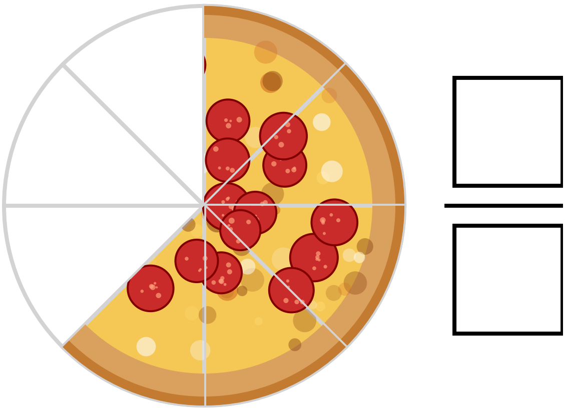


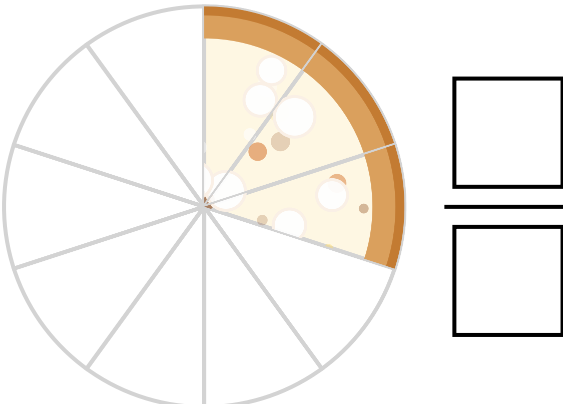


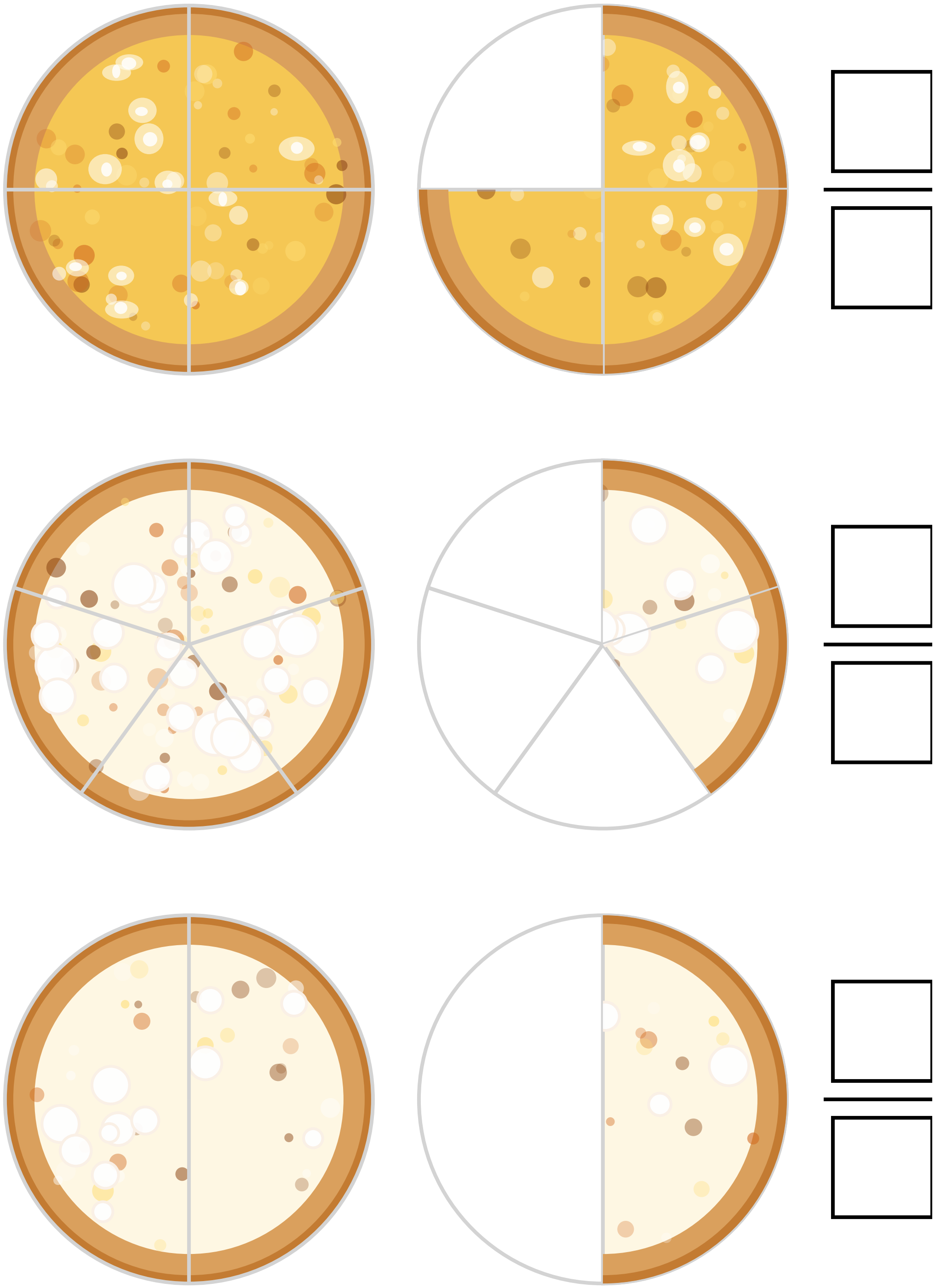








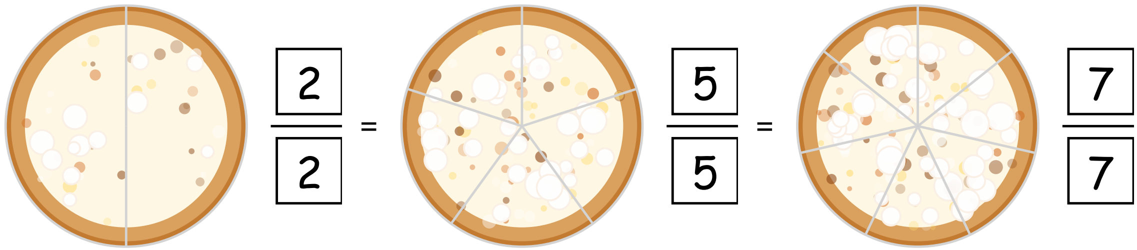




7.3 Comparing fractions

The reason the integers $\mathbb{Z}$ are included in the rational numbers $\mathbb{Q}$ is that the integers can be represented as fractions.

For example, no matter how many slices a pizza is divided into, if you eat all of them, you have eaten 1 whole pizza.



The same applies to any other integer. For example, 2 can be written as:

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \cdots$$

A fraction is equal to an integer whenever the denominator exactly divides the numerator. Here is another example:

$$-3 = \frac{-3}{1} = \frac{-6}{2} = \frac{-9}{3} = \frac{-12}{4} = \cdots$$

Convince yourself that when the numerator and the denominator are equal, the fraction is equal to 1. If the absolute value of the numerator is greater than the absolute value of the denominator, the absolute value of fraction is greater than 1. Finally, if the absolute value of the numerator is less than the absolute value of the denominator, the absolute value of the fraction is less than 1.

Absolute value

The absolute value of a number is the number's value without sign. For example, the absolute value of -3, represented as $|-3|$, is 3, and the absolute value of 3, represented as $|3|$, is also 3.

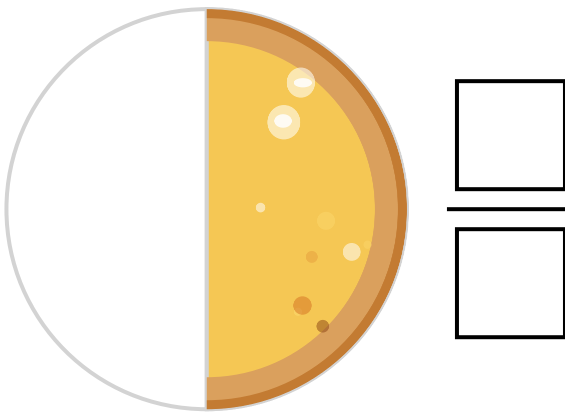
$$\frac{a}{a} = 1 \quad \text{for any } a \neq 0$$

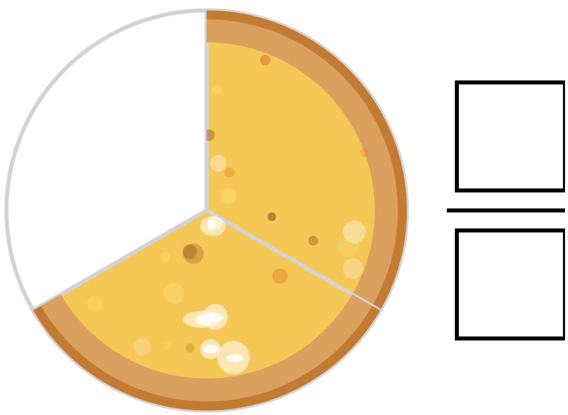
$$|\frac{a}{b}| > 1 \quad \text{if } |a| > |b| \quad \text{and } b \neq 0$$

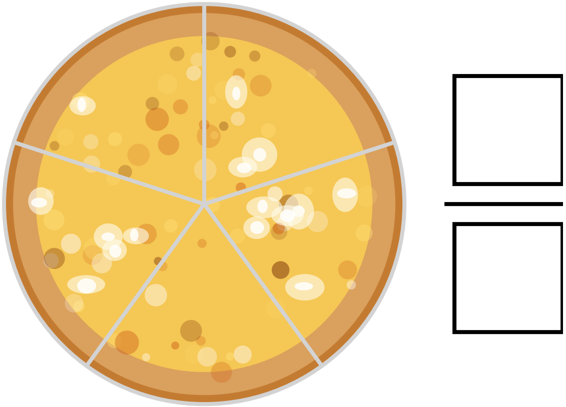
$$|\frac{a}{b}| < 1 \quad \text{if } |a| < |b| \quad \text{and } b \neq 0$$

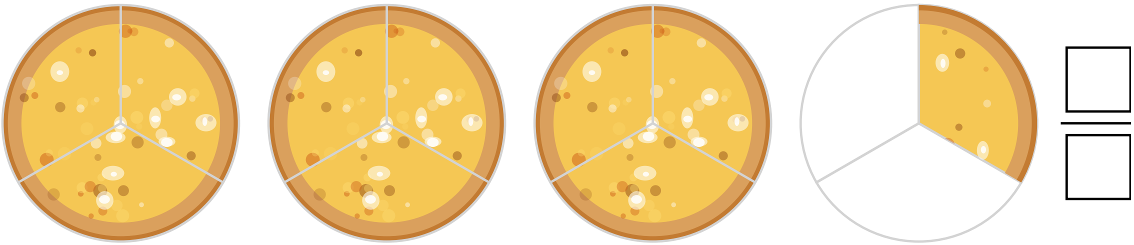
Exercises

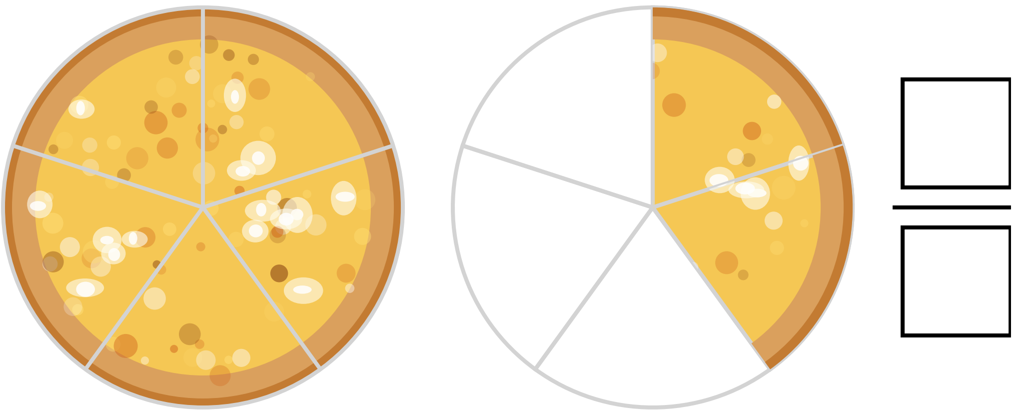
Write down the following fractions and determine whether they are equal to, greater than, or less than 1:

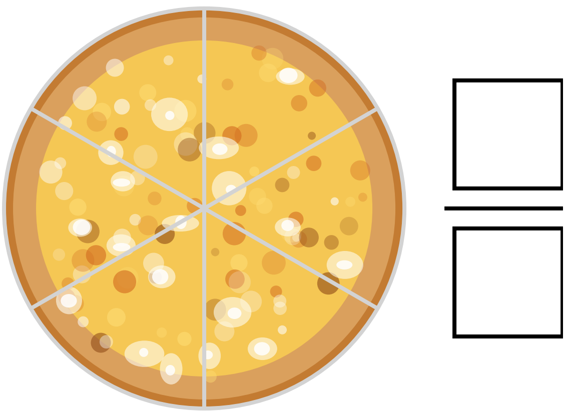


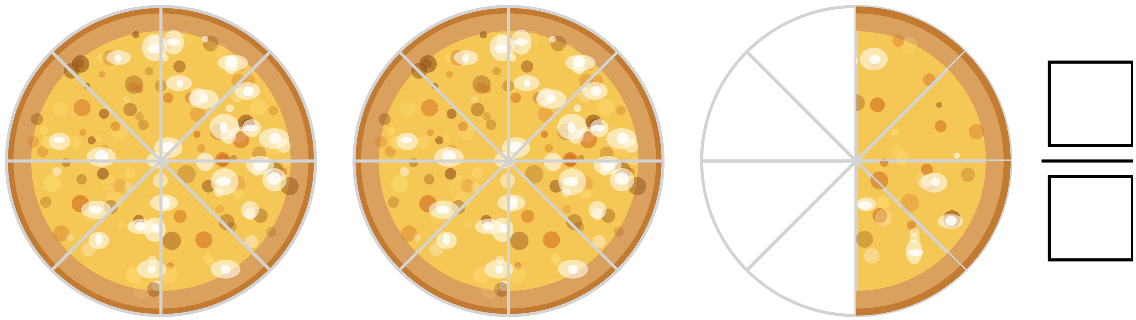




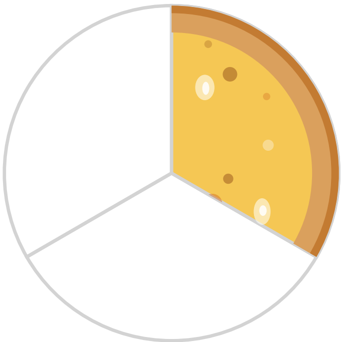
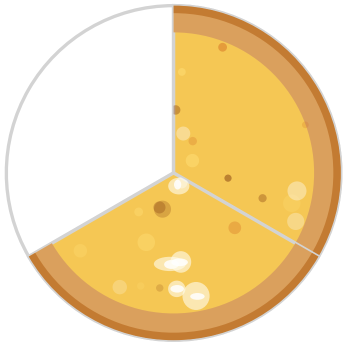


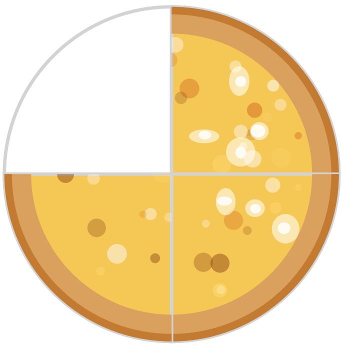
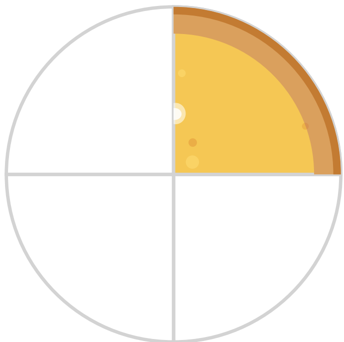


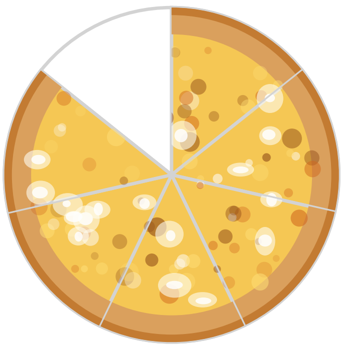
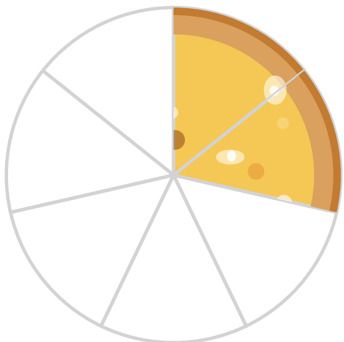


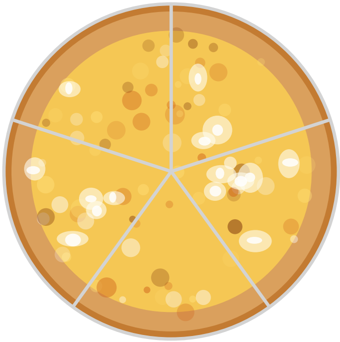
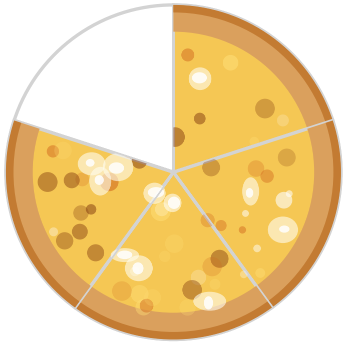


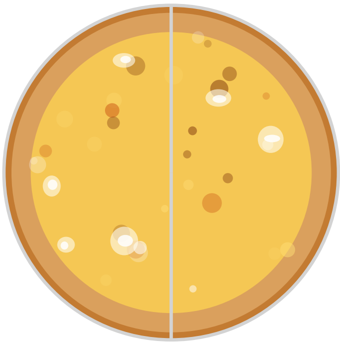
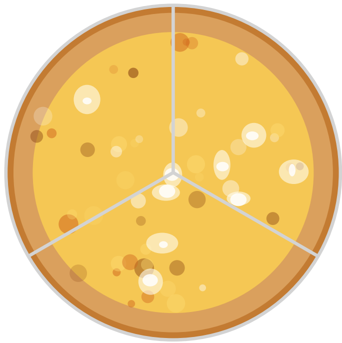
Compare the following quantities and write the appropriate symbol in the empty box: either <, or >, or =.

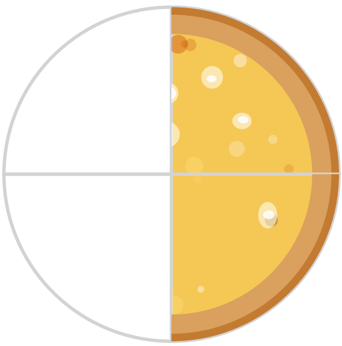
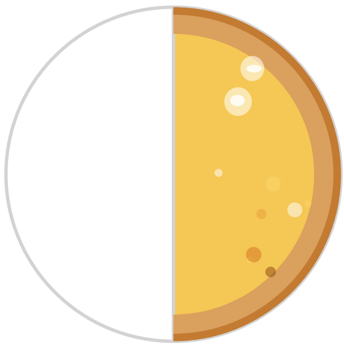
 $\frac{1}{3}$ $\square$  $\frac{2}{3}$

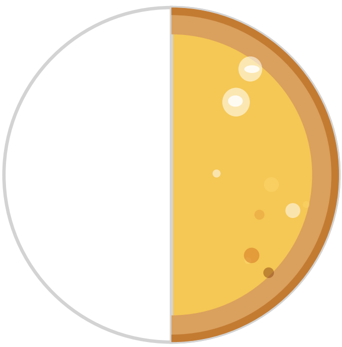
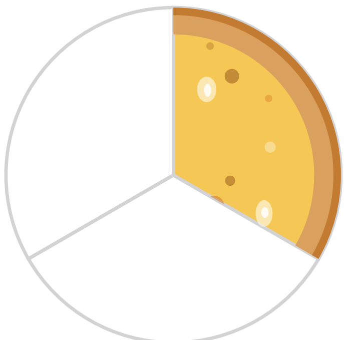
 $\frac{3}{4}$ $\square$  $\frac{1}{4}$

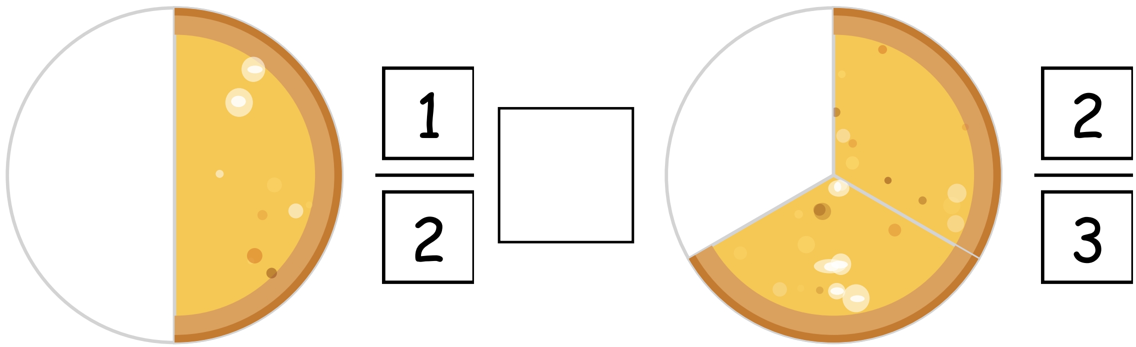
 $\frac{6}{7}$ $\square$  $\frac{2}{7}$

 $\frac{5}{5}$ $\square$  $\frac{4}{5}$

 $\frac{2}{2}$ $\square$  $\frac{3}{3}$

 $\frac{2}{4}$ $\square$  $\frac{1}{2}$

 $\frac{1}{2}$ $\square$  $\frac{1}{3}$



7.4 Fractions make you dance ♫

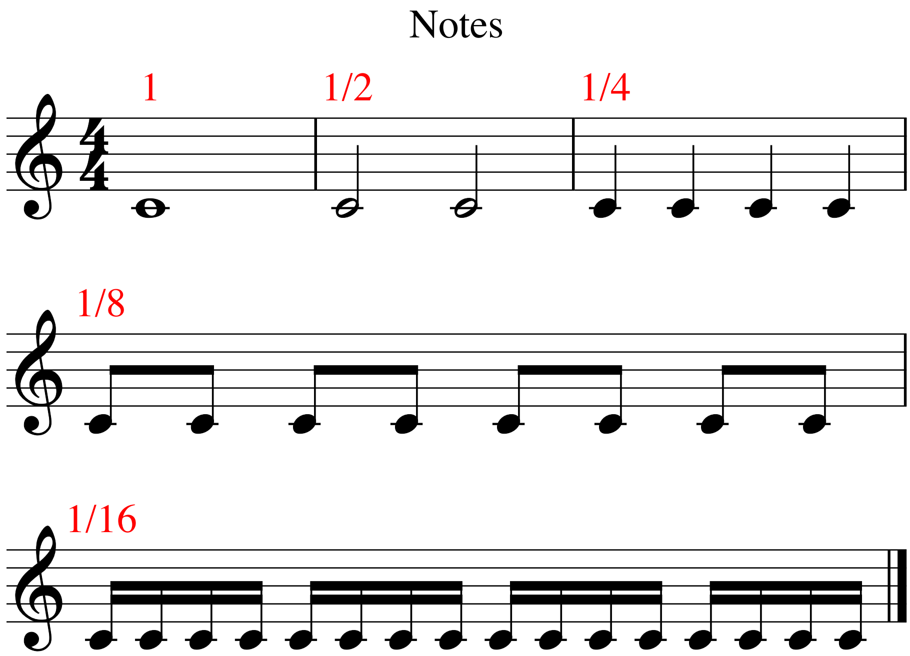
Fractions appear in many contexts, such as nature and art, where they describe proportions, symmetries, and ratios. In music, note durations are expressed as fractions relative to a whole note. What makes you dance when you listen to music is precisely the alternation of notes and rests of different durations. In other words, it is a succession of fractions that makes you dance.

The following symbols represent notes and rests (yes, silence is also very important in music) of different durations.

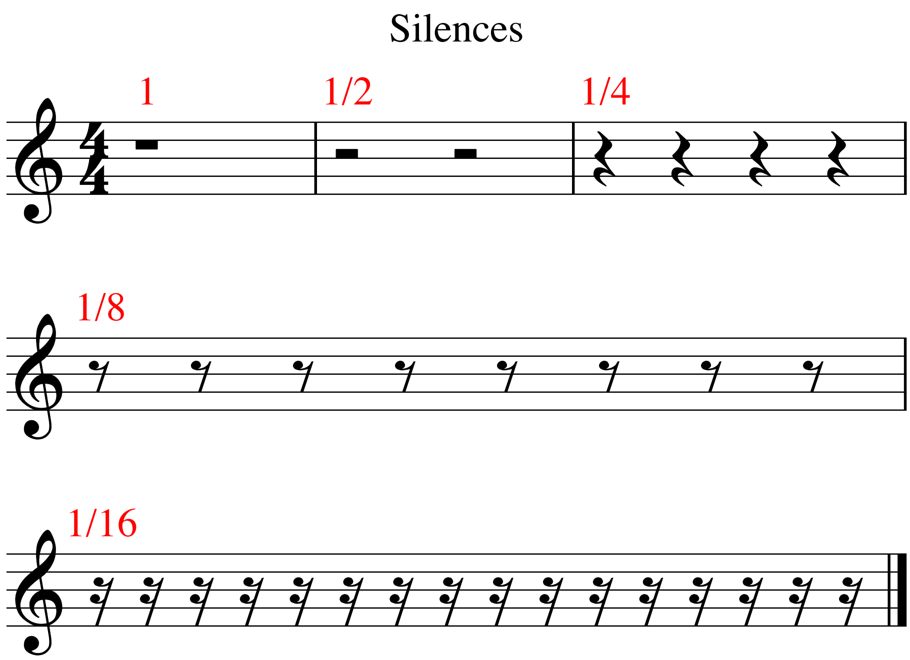
Note	Value	American Name	British Name
𝅜	2	Double whole note	Breve
𝅝	1	Whole note	Semibreve
𝅗𝅥	$\frac{1}{2}$	Half note	Minim
𝅘𝅥	$\frac{1}{4}$	Quarter note	Crotchet
𝅘𝅥𝅮	$\frac{1}{8}$	Eighth note	Quaver
𝅘𝅥𝅯	$\frac{1}{16}$	Sixteenth note	Semiquaver

In the following score, each bar—each division of the staff—contains notes that sum to 1. 1 "whole" note worth 1, 2 half notes each worth $\frac{1}{2}$, 4 quarter notes each worth $\frac{1}{4}$, 8 eight notes each worth $\frac{1}{8}$, and 16 sixteenth notes each worth $\frac{1}{16}$.

> **Do the notes in a bar sum up to 1?**
>
> Add up the values of the notes in each bar to see that they sum to 1.

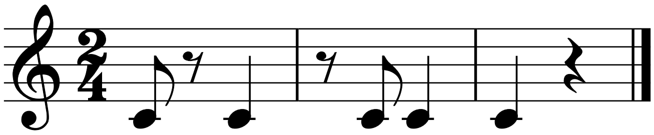


And similarly for the rests:



In the following score, the values in each bar sum up to $\frac{2}{4}$, that is the same as $\frac{1}{2}$.

2/4



8 Sets and Functions

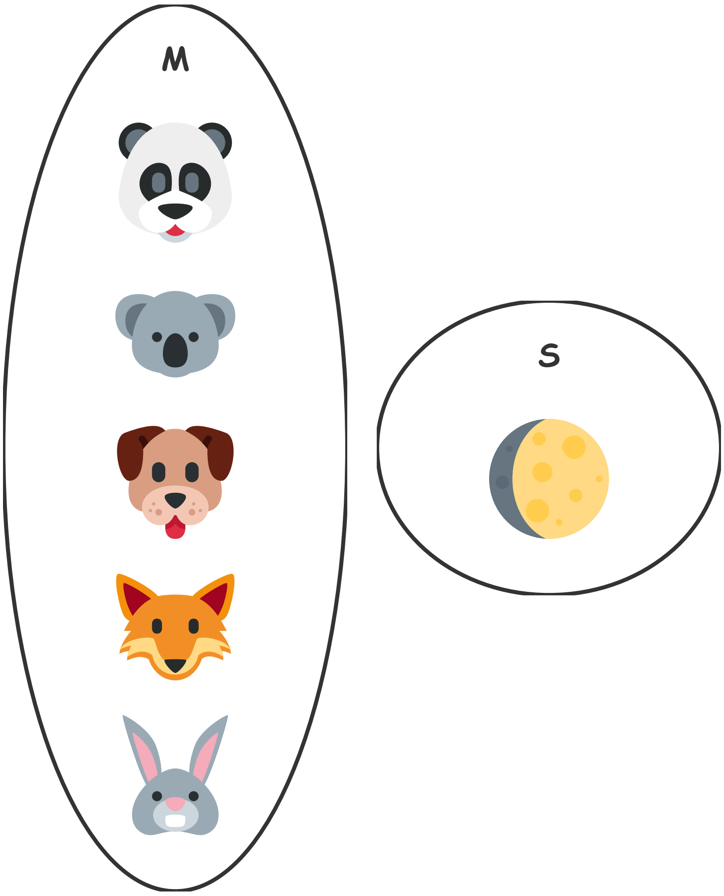
As we saw before, sets are a collection of objects. We have encountered prominent examples like $\mathbb{N}$, the set of counting or natural numbers, $\mathbb{Z}$, the set of integers, positive and negative, and $\mathbb{Q}$, the rational numbers.

We also made up other examples, such as M, a set of mammals, and S, the set of the natural satellites of Earth.

$$M = \{\text{🐨}, \text{🐶}, \text{🐼}, \text{🦊}, \text{🐰}\}$$

$$S = \{\text{🌔}\}.$$

These sets can also be represented with Euler-Venn diagrams.

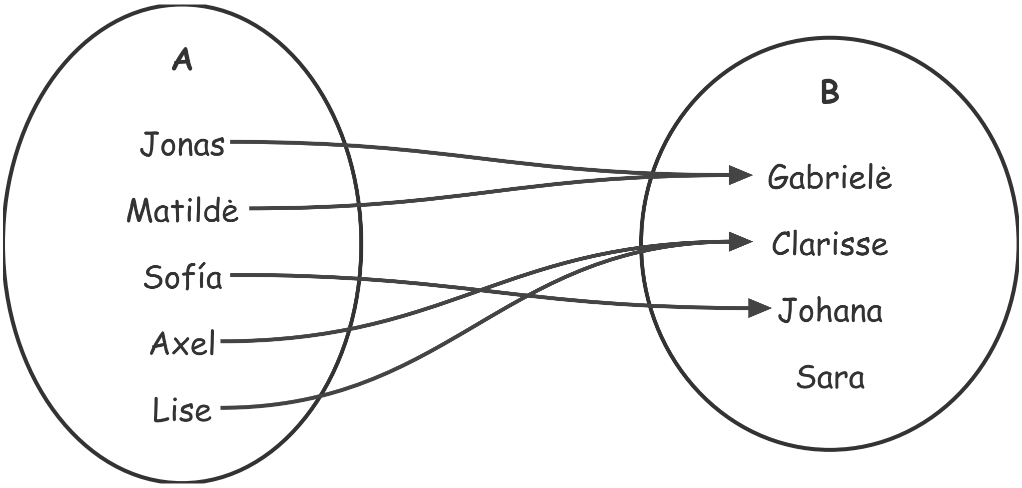


In this chapter, we will learn how to define relationships between sets, including those between a set and itself. We will also introduce the concept of a function, a specific type of relationship between sets.

8.1 Relationships between sets

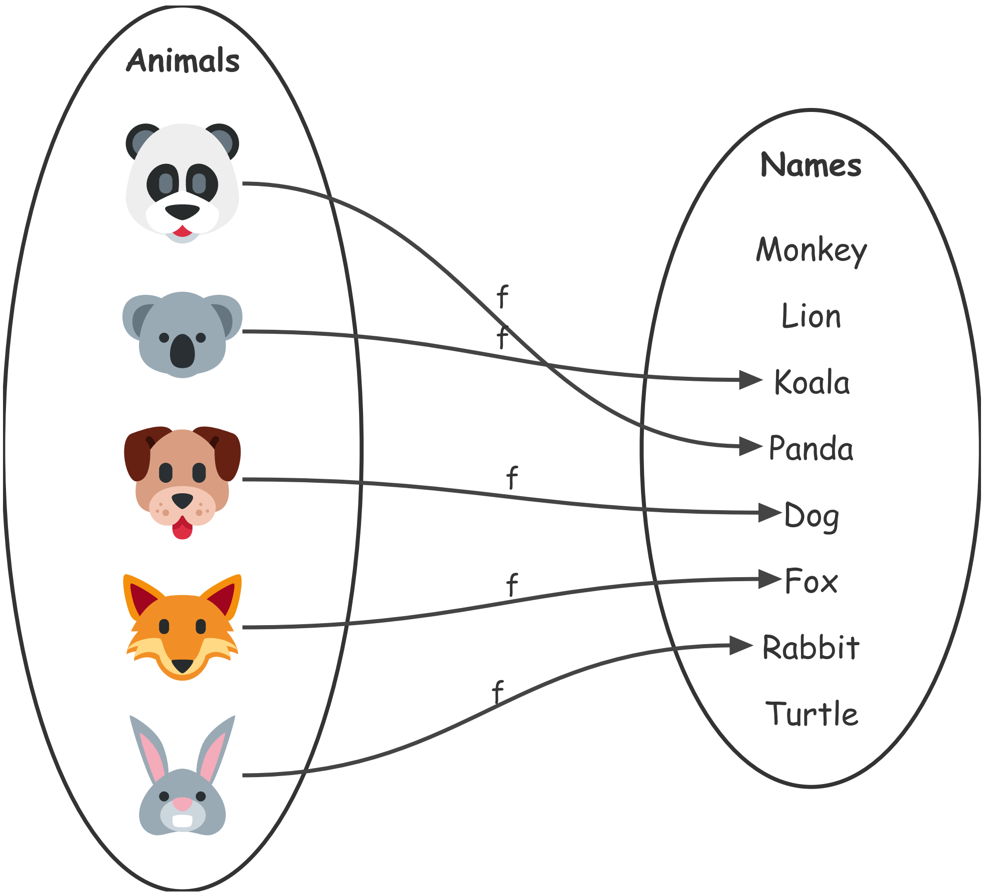
Given two sets, A and B, we can define a mapping, or relationship, that maps or projects elements of A onto elements of B.

For instance, suppose that A is a set of children, such as you and your friends. Set B is a set of their moms. We can define the relationship "being the mom of" 🤱 that maps every child in A onto his/her mom in B.



What can you infer from this mapping?

A **function** is a rule that maps the elements of one set onto the elements of another. In the following example, the function f maps an animal's image to its English name.



We call the set of Animals the *domain* of the function f. The function f :

Animals → Names is the rule that allows us to map an animal image to its English name. For example:

$$f(\text{🐰}) = \text{Rabbit}$$

We say that Rabbit is the *image* of 🐰.

> **Are all mappings between two sets functions?**
>
> No. A relationship is only a function if **every** element in the first set is mapped to **exactly one** image in the second set.

The previous example is a function because every animal in the set Animals is mapped to exactly one Name.

However, imagine that we discover a new animal for which we have no English name yet. The following mapping is **not** a function because at least one element in the domain does not have a known name. The requirement is that **every** element has exactly one.

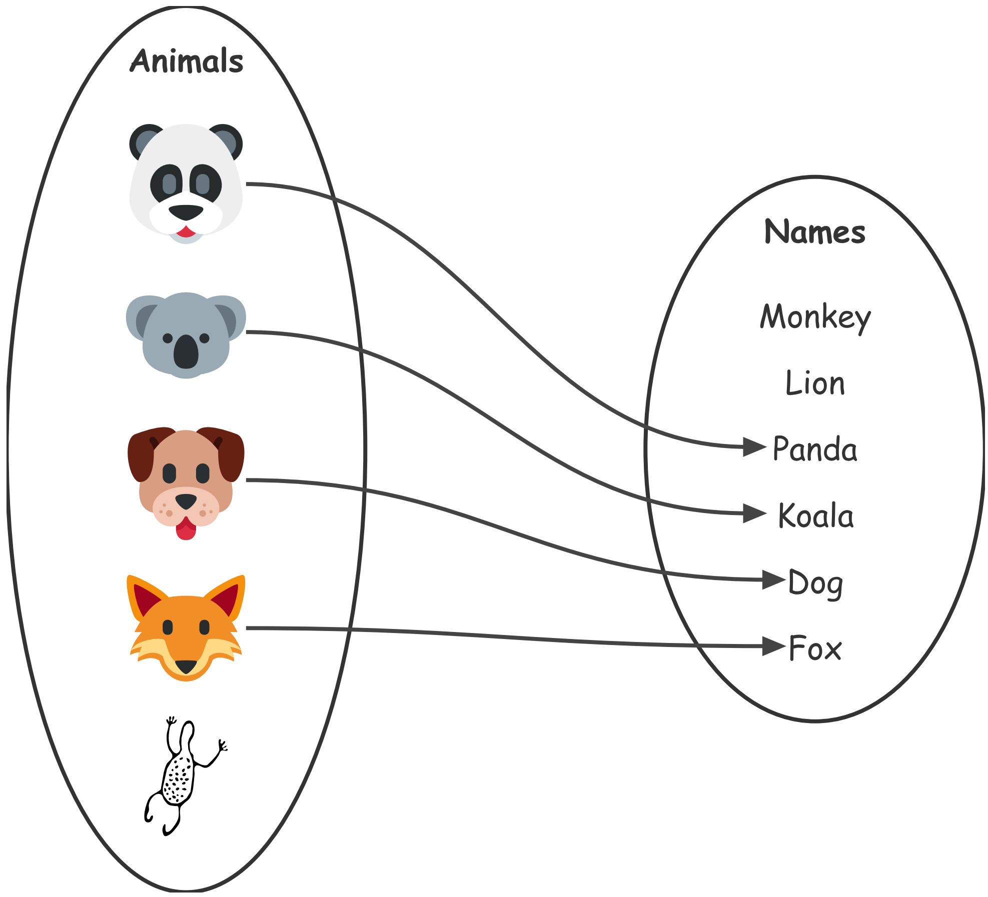


Figure 8.1: This is not a function because one of the animals is unnamed.

Similarly, if an animal has two names, That relationship is also not a function because every element in the domain must have exactly 1 image.

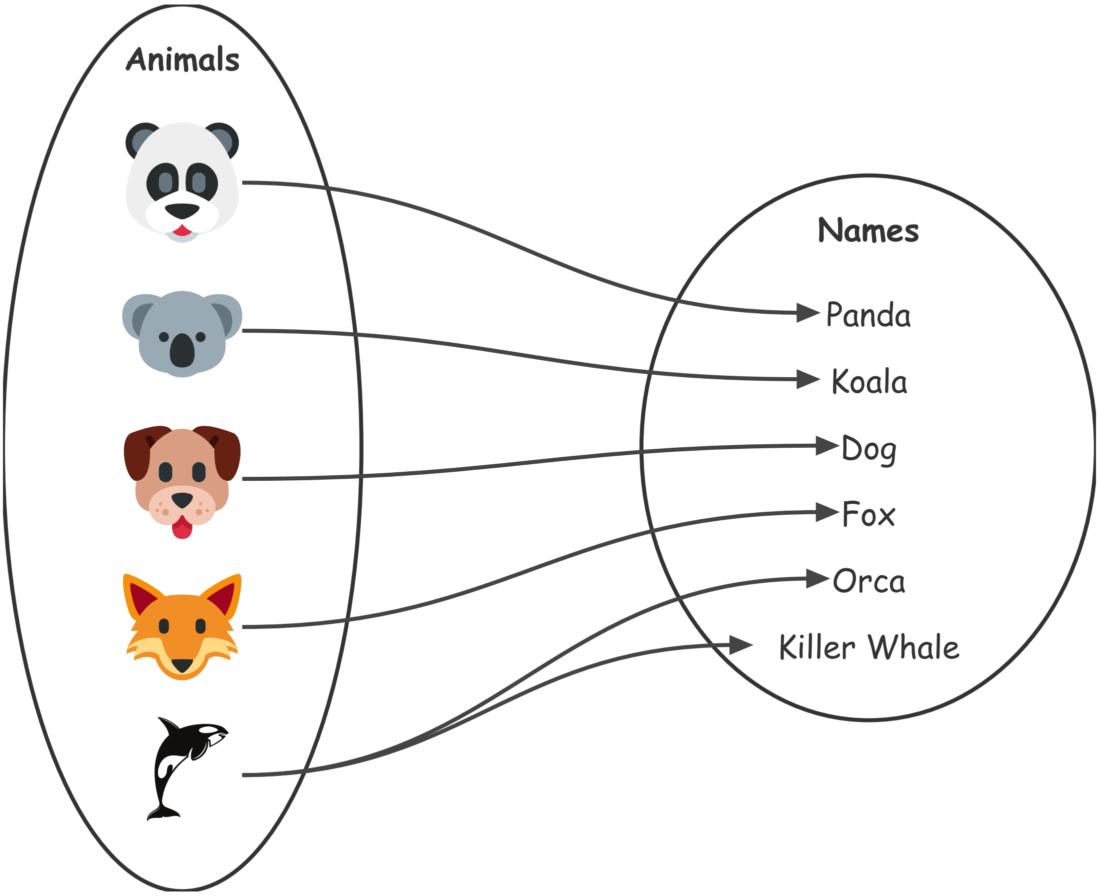
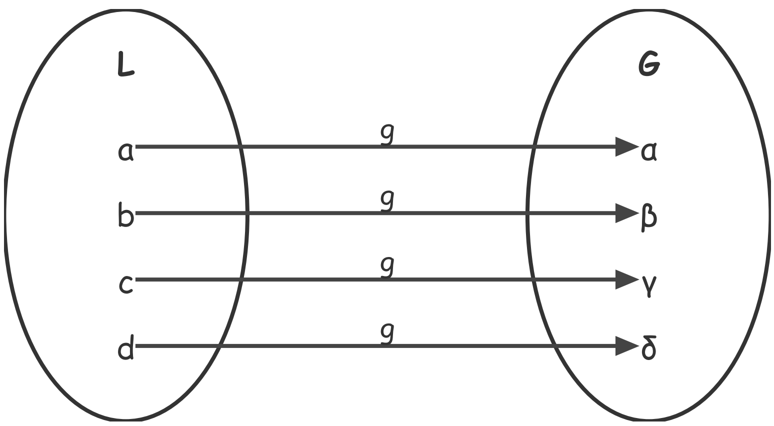


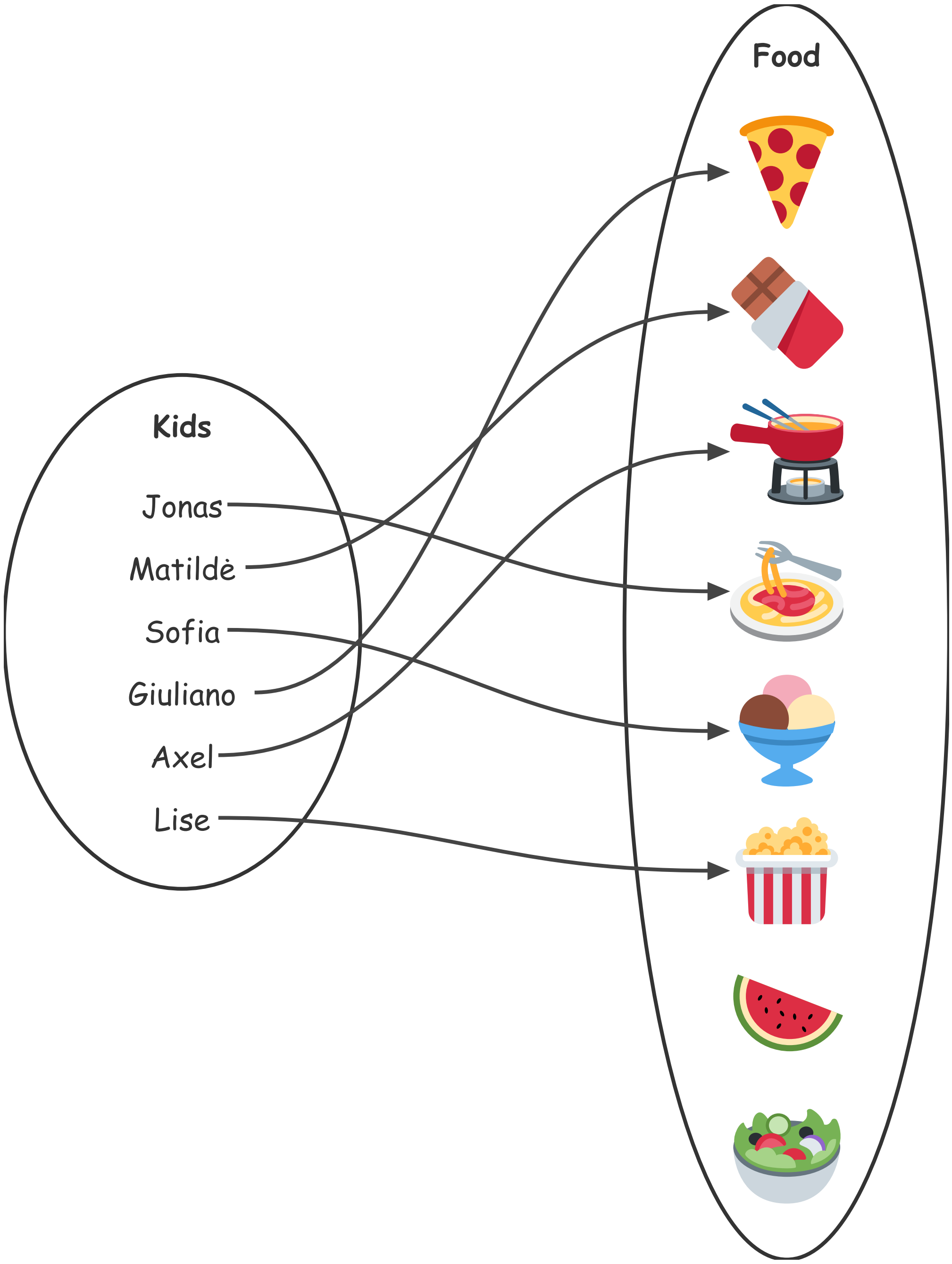
Figure 8.2: It's also not a function because one of the animals has more than one name.

8.2 Exaples of functions

The function $g : L \to G$ maps the Latin alphabet onto the Greek alphabet.

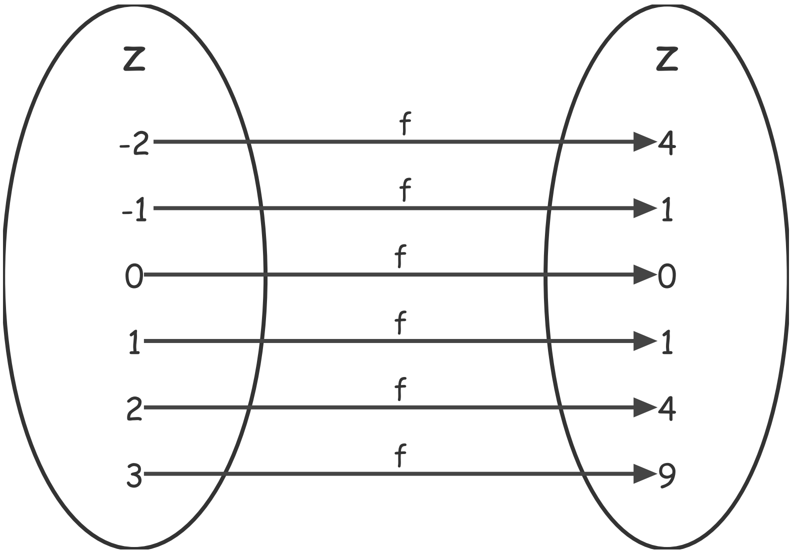


The function h : Kids $\to$ Food maps a set of kids to their favorite food.



Finally, a function can also map a set to itself. For example, consider the function $f : \mathbb{Z} \to \mathbb{Z}$, which maps every integer to its square.

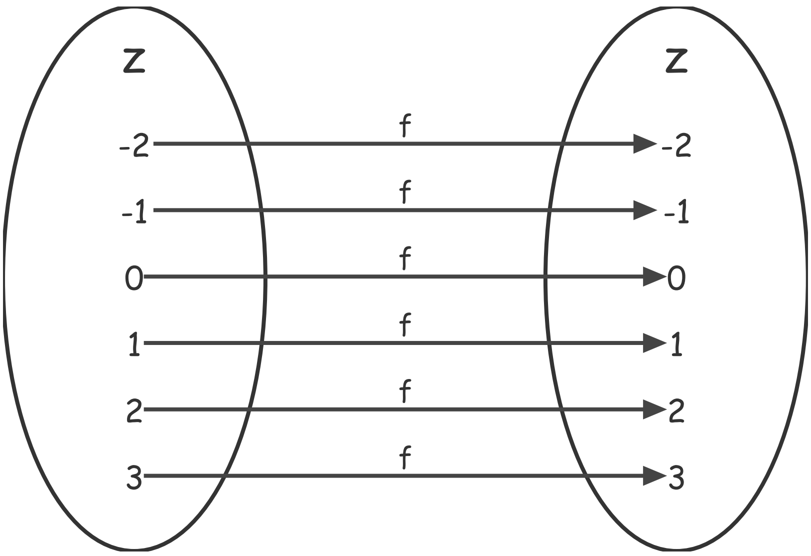
$$f(x) = x^2, \quad \text{for every } x \in \mathbb{Z}$$



A more trivial example is the identify function $f : \mathbb{Z} \to \mathbb{Z}$

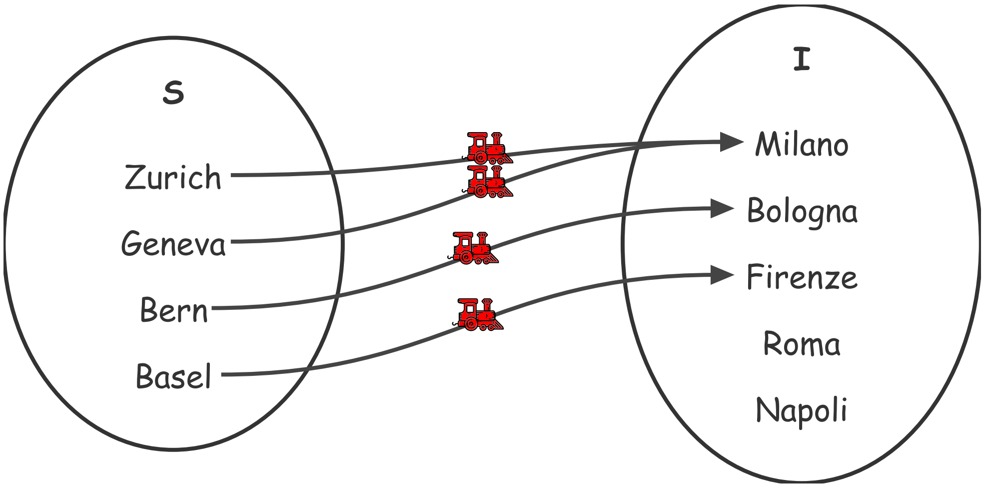
$$\mathrm{f(x)} = \mathrm{x}$$

which maps every integer to itself.



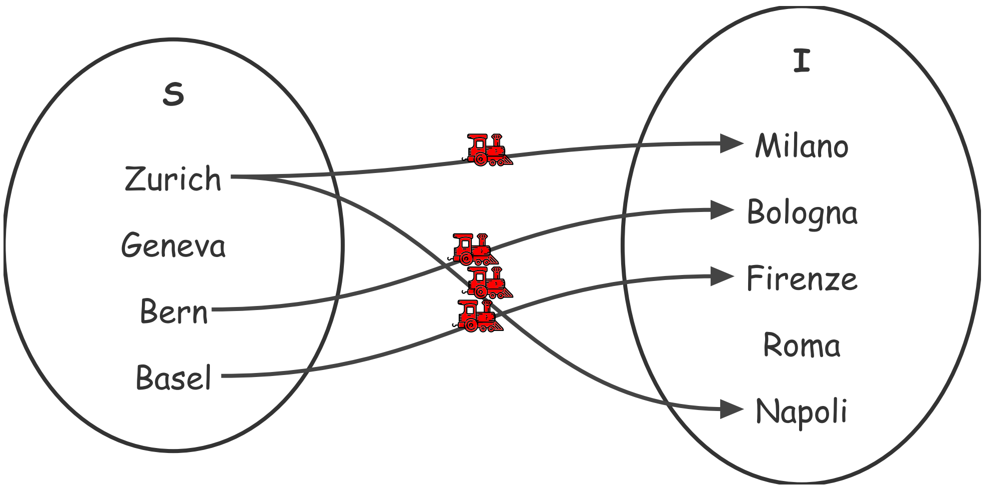
Exercises

The following examples show a mapping that connects cities in Switzerland to cities in Italy. Determine whether this mapping is a function and explain your answer.



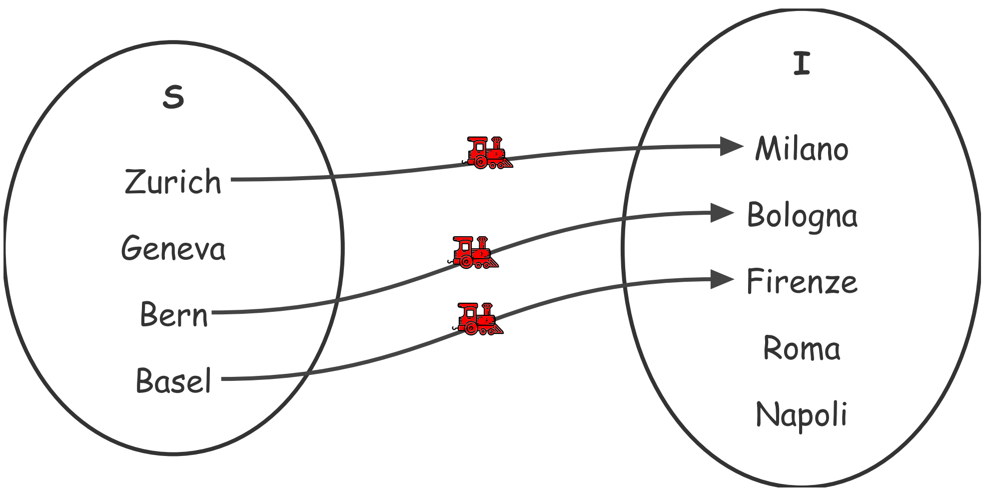
Is the previous example a function?

Why?



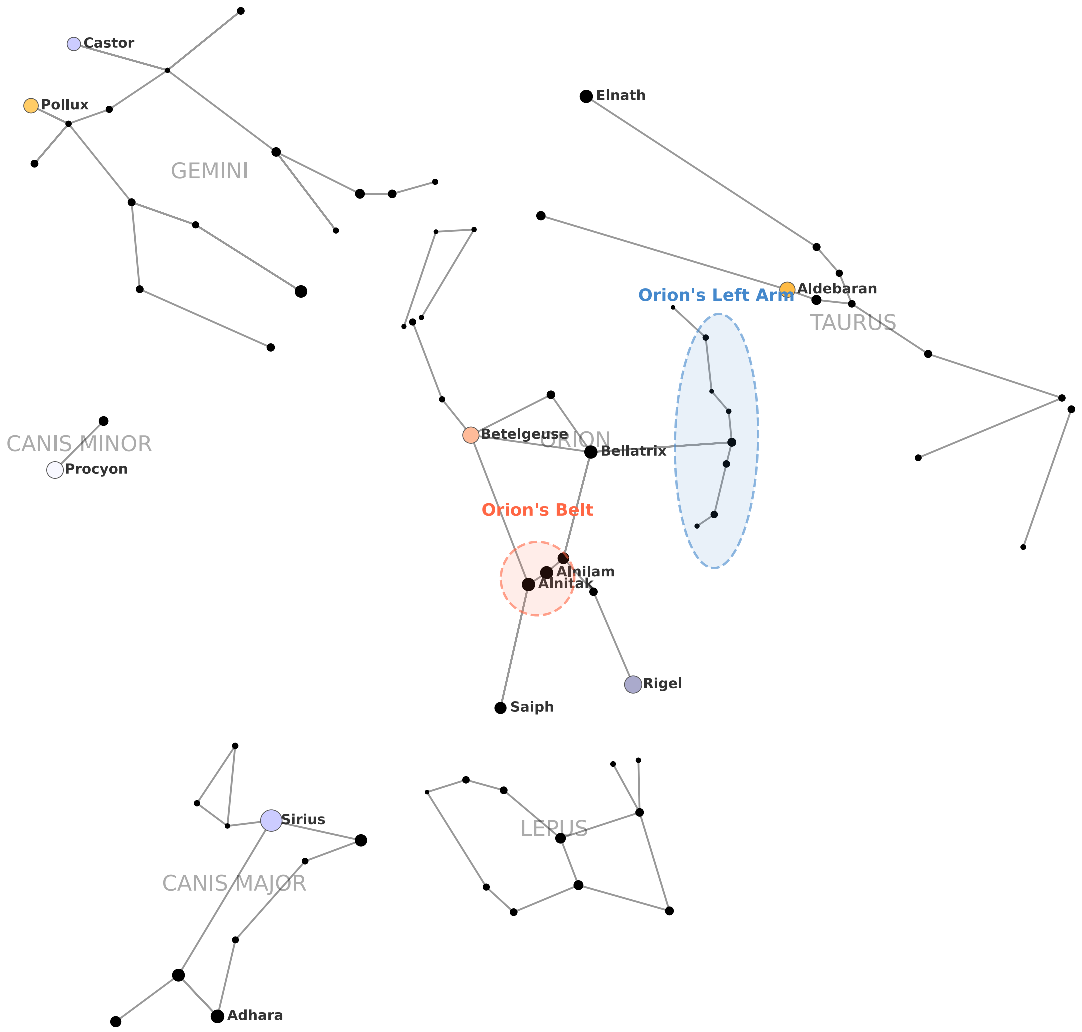
Is the previous example a function?

Why?

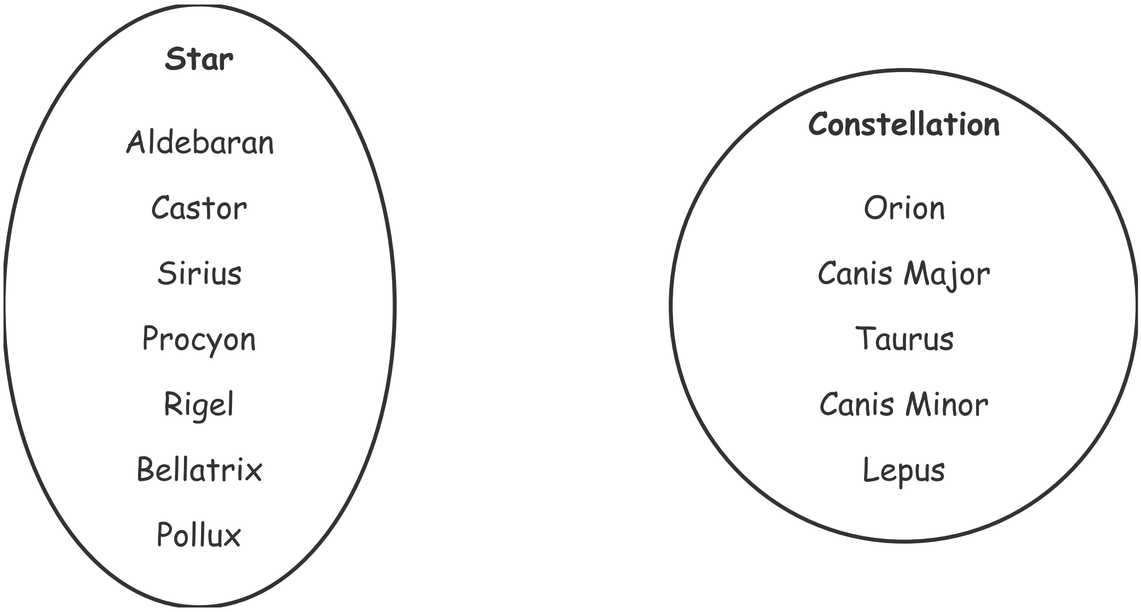


Is the previous example a function?

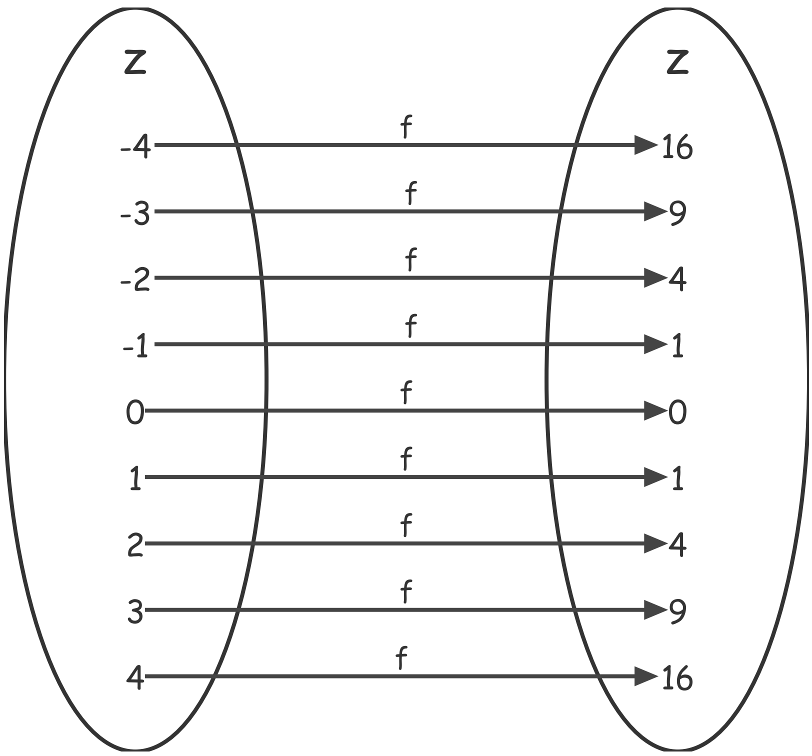
Why?



Let the mapping $c : \text{Star} \to \text{Constellation}$ be the mapping of a star to the constellation to which it belongs. Draw the mapping between the sets Star and Constellation, and explain whether this mapping is a function.

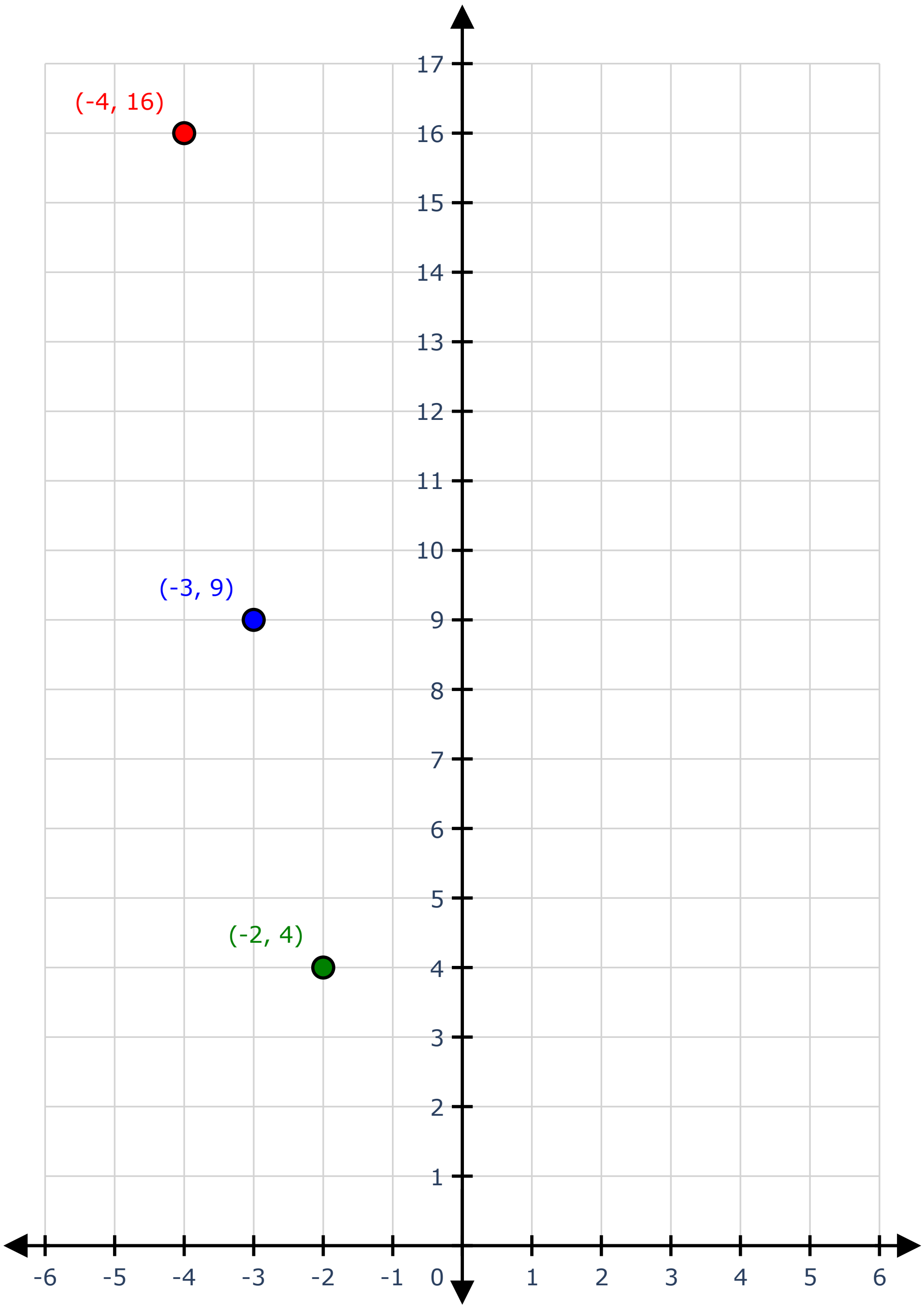


It is possible to represent numerical relationships, such as the function $f : \mathbb{Z} \to \mathbb{Z}$, defined as $f(x) = x^2$, which maps every integer to its square, on a Cartesian plane.



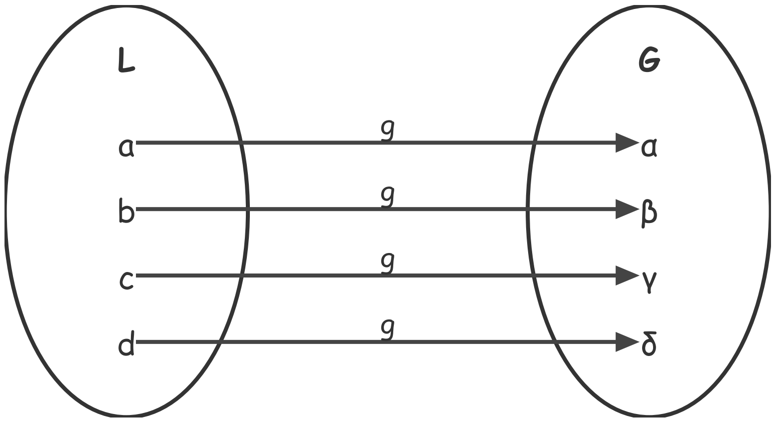
Take every pair of elements: an element in A and its image in B, and locate them on the Cartesian plane as follows: the element in A is the x-coordinate, and the element in B is the y-coordinate.

In the following figure, I added the first three pairs: $(-4, 16)$, $(-3, 9)$, and $(-2, 4)$. Add the rest, and then connect those points with a line in the same order. First, connect $(-4, 16)$ to $(-3, 9)$, then $(-3, 9)$ to $(-2, 4)$, and so on.



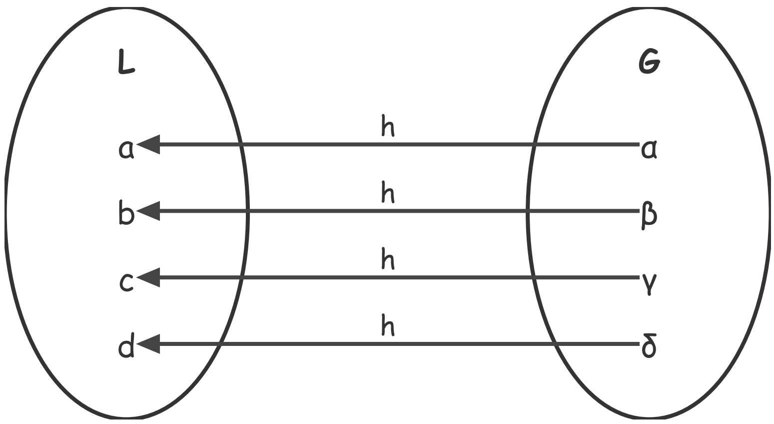
8.3 Inverse of a function

We encountered in the examples a function $g : L \to G$ that maps Latin letters into Greek ones.



Imagine that we want to do the opposite, namely mapping Greek letters into Latin ones.

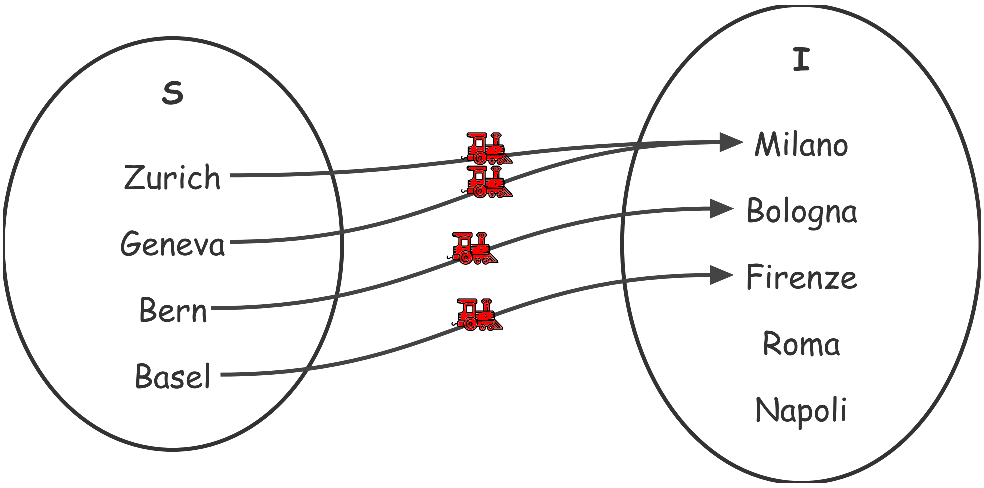
To to this, we need to reverse the mapping. We need a new function, let's call it h, that maps Greek into Latin, $h : G \to L$.



In a way, this function $h : G \to L$ undoes what the function $f : L \to G$ does, and we call h the inverse function of f. We denote the inverse as $h = f^{-1}$.

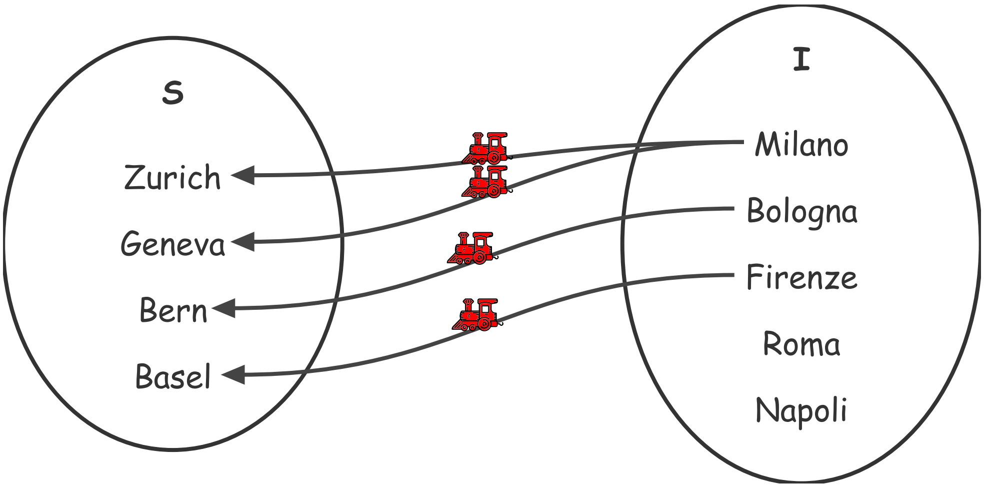
> **Question**
>
> Is it always possible to find an inverse?

Let's consider another example to answer this question. As you saw in the exercises, the following mapping $f : S \to I$ is a function because every element in S has exactly one image in I. In other words, every city in Switzerland is connected to at most one city in Italy.



Unfortunately, the holidays are over, and we have to travel back from Italy to Switzerland. We need a function that goes in the opposite direction. This function is called the inverse of f, and we use the symbol f^{-1} for it.

We need a new function $g : I \to S = f^{-1}$, that undoes $f : S \to I$.



Is this mapping a function?

The departure set, or domain, is Italy, or I.

- Does *every* element in I have an image in S? Does a train leave from every city in Italy?
- Does *every* element in I have a unique image in S? Is each city in Italy connected to exactly one city in Swizerland?

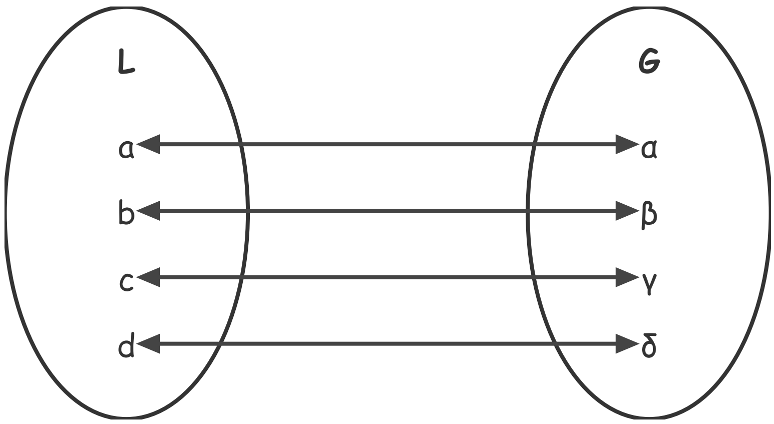
We see that the answer to both questions is no. There are no trains leaving from Rome and Napoli, and Milano is connected to more than one city in Switzerland. The mapping $g : I \to S$ is **not** a function. Therefore, the function $f : S \to I$ is not invertible.

For a function to be invertible, an additional condition is needed.

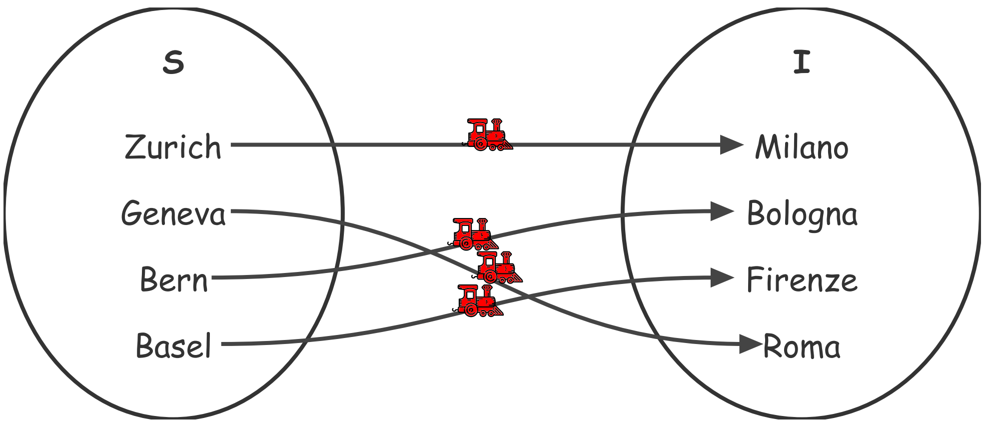
Invertible functions

A function between two sets A and B, $f : A \to B$, is invertible if every element in B is the image of exactly one element in A. In other words, the function must implement a one-to-one correspondence between the two sets. This means that the inverse mapping, $f' : B \to A$, is also a function.

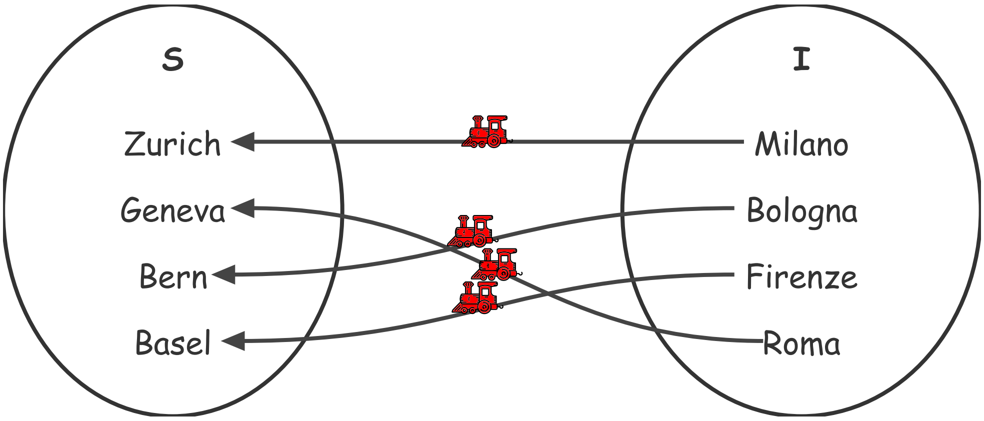
As we saw with the example of mapping Latin letters to Greek letters, this condition is met and the function is therefore invertible.



We can fix the function $f : S \to I$ to be invertible.



This new function is invertible, meaning that we can always return home from Italy by taking the train in the opposite direction.



Question

How did we fix this function?

If we take a train from Zurich, we know that we will arrive at f(Zurich), which is Milan. To return to Zurich, we need to take a train to f^{-1}(Milan), which is Zurich. If we take a train from Zurich, we know we will arrive at $f(\text{Zurich}) = \text{Milano}$. To return to Zurich, we need to take a train to $f^{-1}(\text{Milano}) = \text{Zurich}$.

Exercises

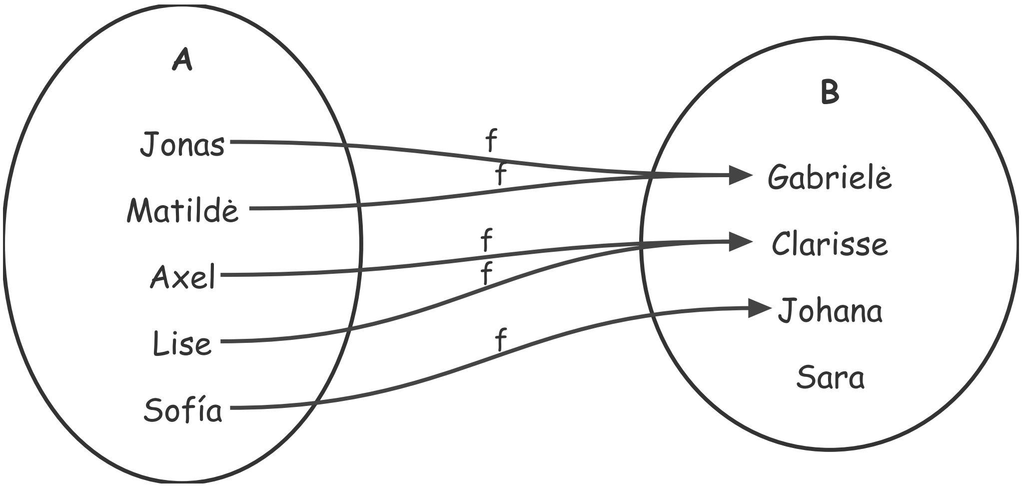
Try reasoning about the inverse function of each of the following functions.

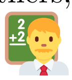
$$f(x) = 2 + x$$

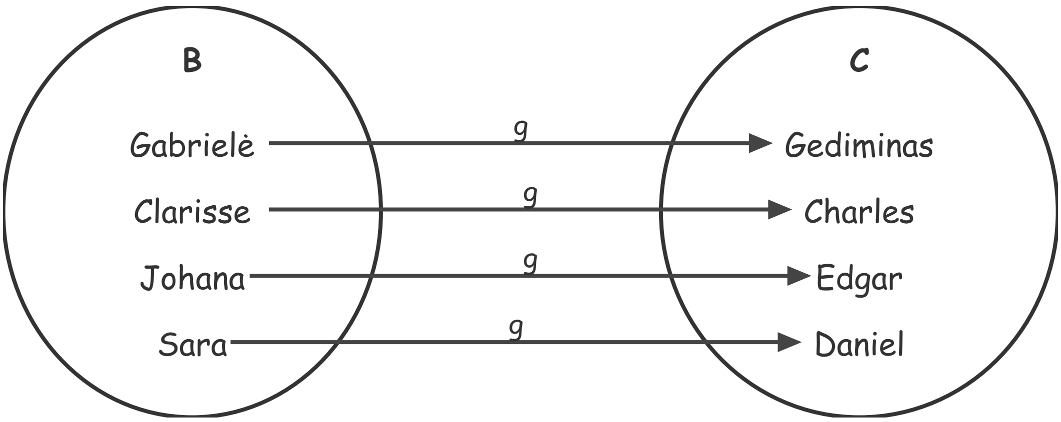
$$f(x) = 2x$$

8.4 Composition of functions

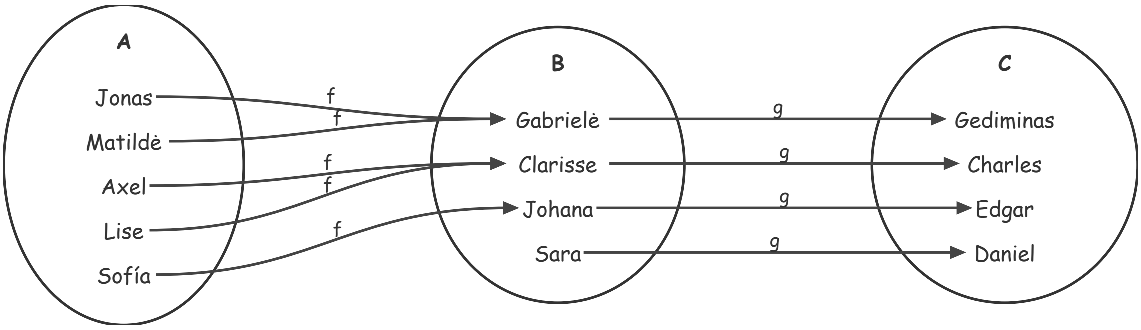
We saw the example of the function $f : A \to B$, which is defined as "being the mom of"  , and maps an element in A (a child) onto his/her mom.



Now, if we introduce a new set, C, consisting of the mother's fathers, we can define a new function $g : B \to C$ as the function "being the father of"  that maps an element in B to her father in C.



We can compose the two functions, meaning we can apply them one after the other.



The composition of the functions is denoted by $[g \circ f](x) = g(f(x))$.

This means that we first apply $f : A \to B$, which sends elements in A to their moms. For example, it sends Jonas $\to$ Gabrielė, because $f(\text{Jonas}) = \text{Gabrielė}$. Then, $g : B \to C$ sends elements in B to their fathers. For example, it sends Gabrielė $\to$ Gediminas, because $g(\text{Gabrielė}) = \text{Gediminas}$.

Question

What relationship does the new function $g(f(x))$ represent?

$$\begin{tikzcd} A \arrow[r, "\text{mom}"] \arrow[rr, bend left, "\text{grandpa}"] & B \arrow[r, "\text{dad}"] & C \end{tikzcd}$$

As a final example, consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = 2x$, which sends an element to its double, and $g : \mathbb{Z} \to \mathbb{Z}$, defined as $g(x) = x + 1$, which increments a number by 1.

Here are some examples:

$$\begin{aligned} f(3) &= 2 \times 3 && = 6 \\ f(-2) &= 2 \times -2 && = 4 \\ f(1) &= 2 \times 1 && = 2 \\ g(1) &= 1 + 1 && = 2 \\ g(0) &= 0 + 1 && = 1 \end{aligned}$$

$$g(\color{red}{-1}\color{black}) = \color{red}{-1}\color{black} + 1 \qquad = 0$$

The composition $[f \circ g](x) = f(g(x))$ will look like this:

$$[f \circ g](3) = f(g(3)) \qquad = 2 \times (3 + 1) = 8$$
$$[f \circ g](-1) = f(g(-1)) \qquad = 2 \times (-1 + 1) = 0$$
$$[f \circ g](1) = f(g(1)) \qquad = 2 \times (1 + 1) = 0$$

The composition $[g \circ f](x) = g(f(x))$ (they are not commutative) will look like this:

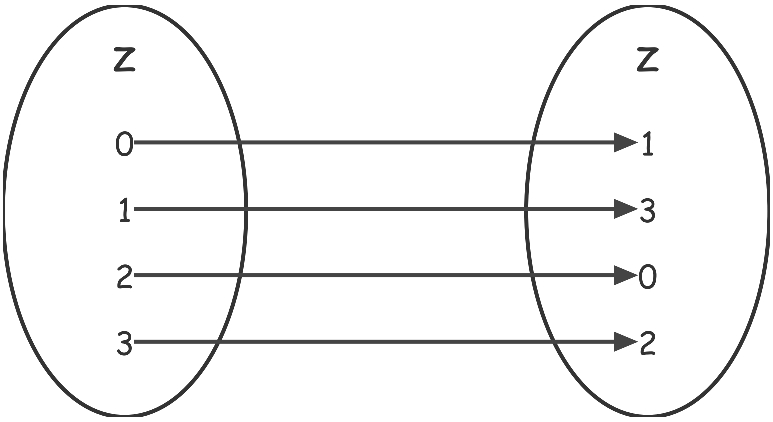
$$[g \circ f](3) = g(f(3)) \qquad = (2 \times 3) + 1 = 7$$
$$[g \circ f](-1) = g(f(-1)) \qquad = (2 \times -1) + 1 = -1$$
$$[g \circ f](1) = g(f(1)) \qquad = (2 \times 1) + 1 = 3$$

8.5 Permutations and Symmetries

Imagine that you have a set of unique game cards representing the values $0, 1, 2, 3$. If you shuffle the cards, they may end up in a different order. For example, they could end up in the order $2, 1, 0, 3$. We call every reordering of the cards a **permutation**.

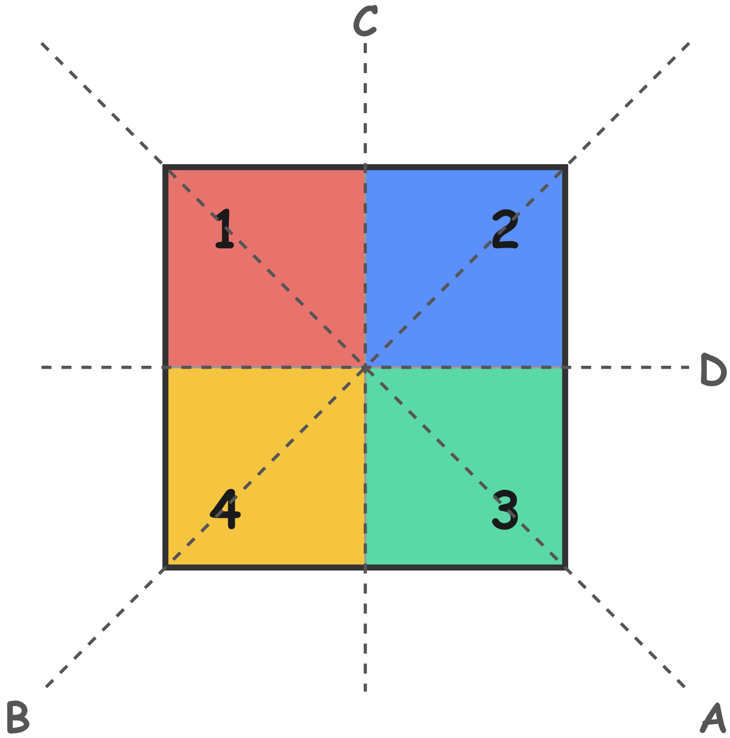
A permutation can be seen as a function—specifically, a one-to-one correspondence, also called a "bijection," or, in other words, an invertible function that maps a set onto itself.

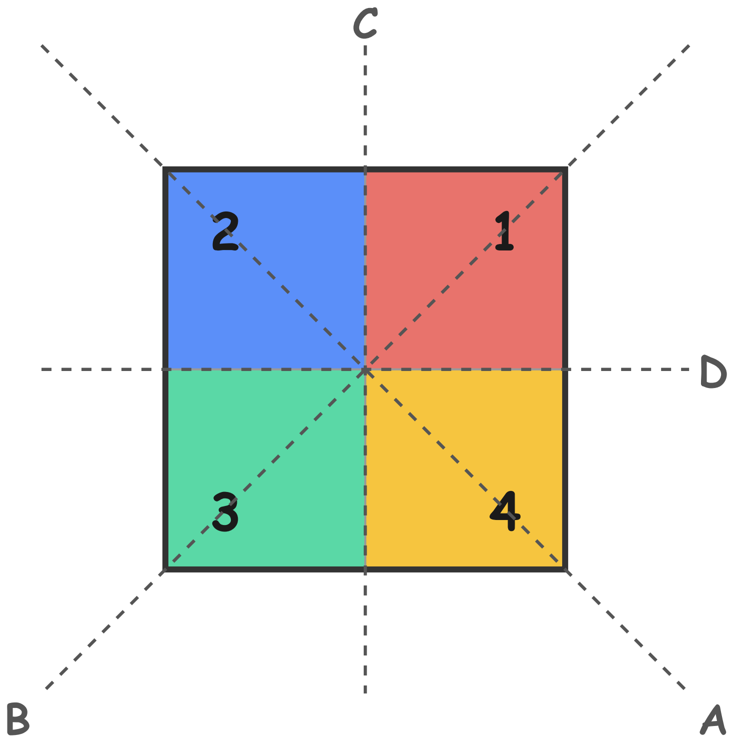
For example, if we have the set $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, an invertible function $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ that reorders the values is called a permutation.

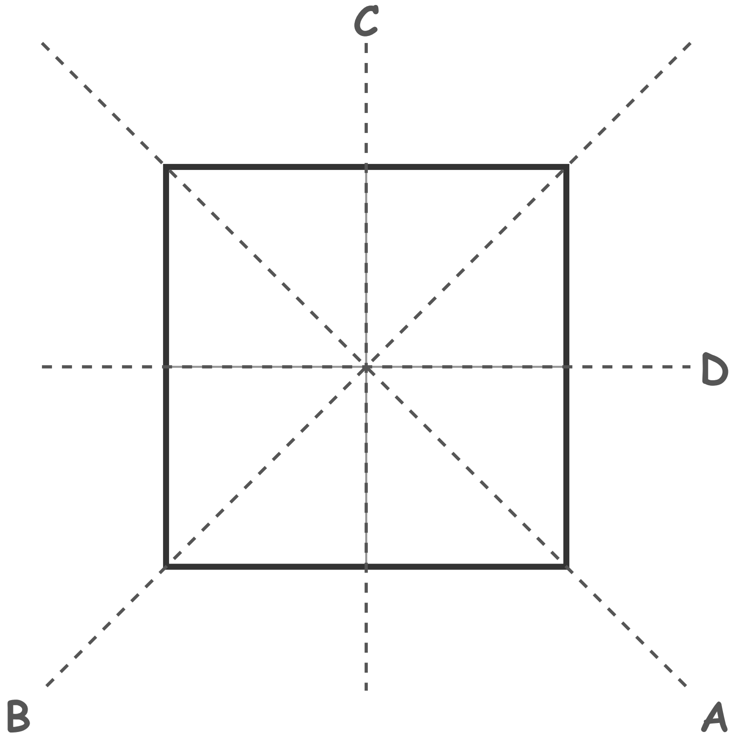


8.6 Symmetries of a Square

Let's do some bricolage! Print this page, then cut out the colored squares. Then, glue them together so that the colors match. It's helpful to print two sets so you can make comparisons. Leave the white square on the page.


C
1
2
D
4
3
B
A


C
2
1
D
3
4
B
A



We will now consider shape-preserving or symmetric transformations of this colorful paper square. In other words, this means any rotation or flipping of the square that allows you to superimpose it on the white square on the page without it falling off.

Parent/educator Tip

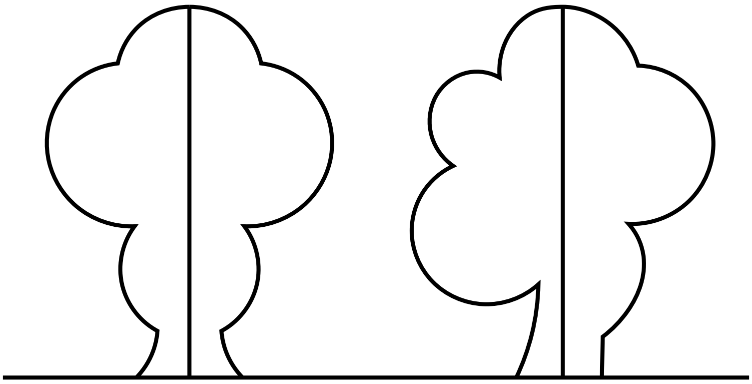
Explain why rotating the square by 45 degrees results in a square that doesn't fit the original or coincide with the fixed white square left on the page.

Symmetry

Symmetry, from the ancient Greek word συμμετρία, is one of the most profound and pervasive concepts in nature and mathematics. It governs our perception of beauty, dictates how energy is conserved, and underpins our understanding of music, physics, chemistry, and the world itself. Symmetry is beautiful and important. In a sense, group theory is the mathematical study of symmetry.

Below are examples of symmetric and asymmetric figures. Source: Wikimedia.

Print this page and cut out the images. Then, fold each image along the vertical, straight line. What is the difference between symmetric and asymmetric figures?

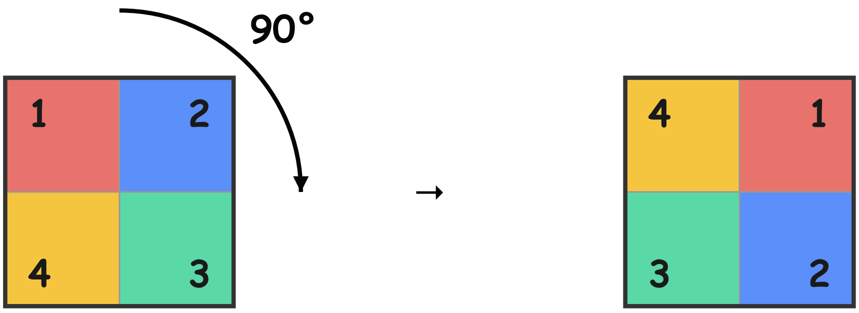


Rotations

One such transformation, for example, is obtained by rotating the square by a quarter ($\frac{1}{4}$) turn (90 degrees).

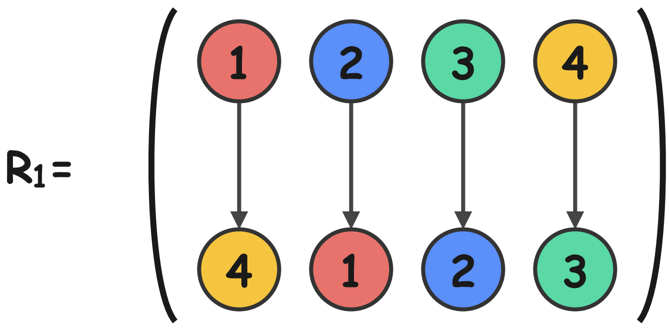
Parent/educator Tip

Here, I am using a passive convention, which means that when we rotate 90 degrees, for example, the set of numbers $\{1, 2, 3, 4\}$ is sent to the set of numbers $\{4, 1, 2, 3\}$. In other words, 1 is mapped to the number that replaces it after the square is rotated while keeping the frame of reference constant. I adopted this convention to make the colors work. It differs from the active convention used in textbooks such as Charles Pinter's wonderful book on abstract algebra, in which rotating by 90 degrees clockwise sends the set of numbers $\{1, 2, 3, 4\}$ to the set of numbers $\{2, 3, 4, 1\}$. In this case, the corner numbers remain fixed, and 1 occupies the position previously occupied by 2. The two approaches should be equivalent, even if some of the rotations are reverted.



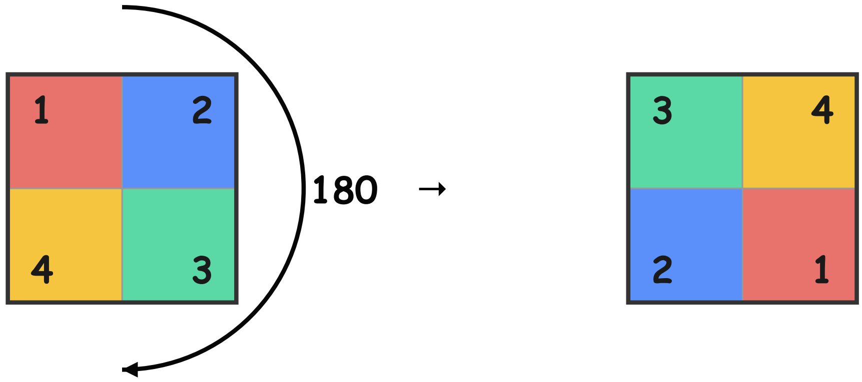
These symmetry-preserving transformations can be thought of as functions that map the set of corners of the square, $S = \{1, 2, 3, 4\}$, to itself. It is a permutation, $R : S \to S$.

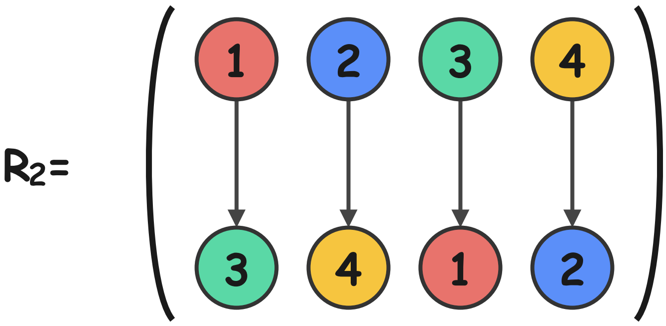
Rotating the square by one-quarter turn can be seen as a function, $R_1 : S \to S$, that sends the original order, $1, 2, 3, 4$, to $4, 1, 2, 3$.



After applying the rotation R_1, 4 is in the position that 1 was in before, 1 is in the position that 2 was in before, and so on.

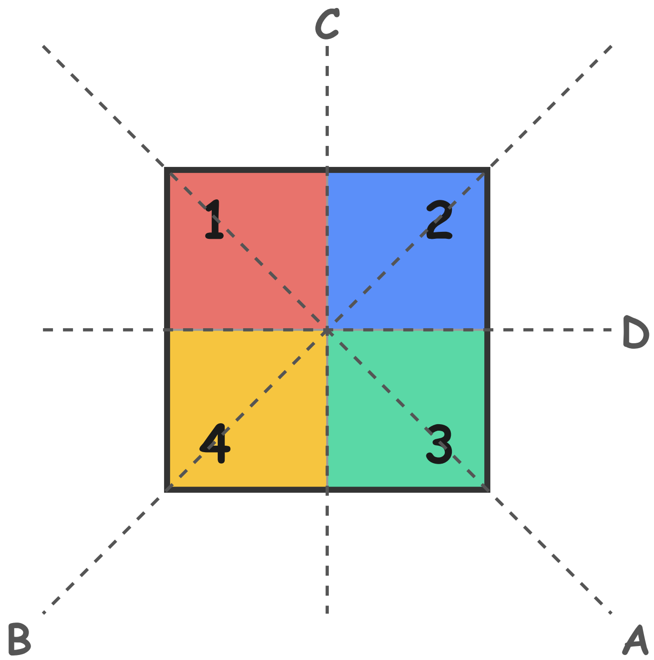
Symmetry is also preserved if we rotate the square by either half (180 degrees) or three-quarters (270 degrees). For example, rotating the square 180 degrees can be seen as a function, $R_2 : S \to S$, that maps the original order, $\{1, 2, 3, 4\}$, to $\{3, 4, 1, 2\}$.





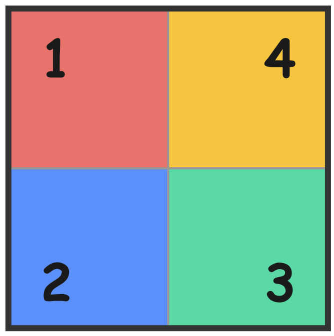
Flips

In addition to rotating the square, we can also flip it around the vertical, horizontal, or diagonal axes. This flipping preserves symmetry, meaning you can still superimpose it on the white square above. Try it yourself!

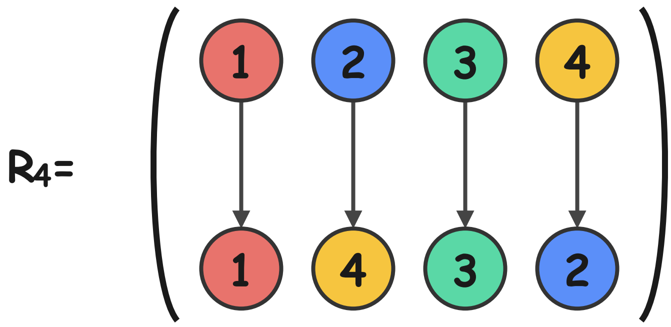


Again, flipping the square along one of its axes can be thought of as a function, $R : S \to S$, or a permutation.

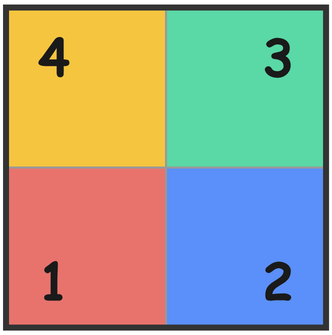
For example, flipping the square along the axis A results in:



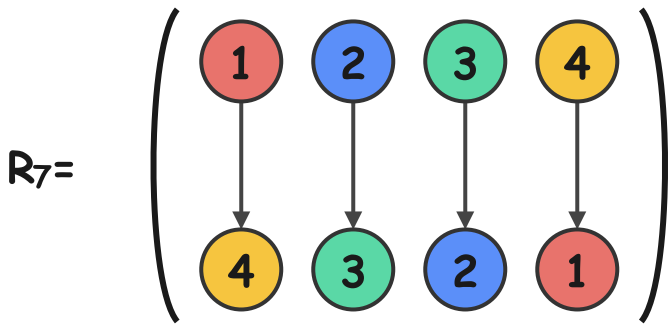
This corresponds to the following mapping:



Another example of a flip is the one around the D axis.



This corresponds to the following mapping:



Here is a summary of the symmetries of our colorful square.

$f : S \to S$	Description
R_0	Identity: do nothing.
R_1	Rotate by 1/4 of a turn, 90 degrees.
R_2	Rotate by 1/2 of a turn, 180 degrees.
R_3	Rotate by 1/4 of a turn, 270 degrees.
R_4	Flip about axis A
R_5	Flip about axis B
R_6	Flip about axis C
R_7	Flip about axis D

Since these transformations are invertible functions, they can be composed. Composition simply means performing one operation after another.

For example, take the paper square and apply the transformation.

$$[R_4 \circ R_1](x) = \overset{\text{Then flip about axis } A}{R_4}\Big(\overset{\text{First, rotate by 90 degrees}}{R_1(x)}\Big)$$

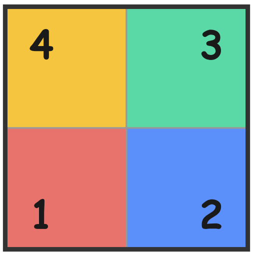
First, we apply R_1, which rotates the square by 1/4. Then, we apply R_4, which flips about the axis A, to the result.

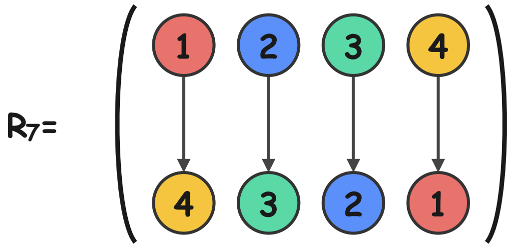
$$R_4 \circ R_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 1 \end{pmatrix} = R_7$$

> **Question**
>
> What does it mean that $R_4 \circ R_1 = R_7$?

It means that applying $R_4 \circ R_1$, i.e., first rotating by 90 degrees and then flipping about axis A, is equivalent to applying R_7, which flips the square about axis D.

Try to convince yourself.





Group of symmetries

The symmetries of the square, $S_G = \{R_0, R_1, \cdots R_7\}$, forms a group under the operation of function composition, $\circ$. Remember our checklist? Let's take a look at the Cayley table of the group.

$\circ$	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7
R_0	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7
R_1	R_1	R_2	R_3	R_0	R_6	R_7	R_5	R_4
R_2	R_2	R_3	R_0	R_1	R_5	R_4	R_7	R_6
R_3	R_3	R_0	R_1	R_2	R_7	R_6	R_4	R_5
R_4	R_4	R_7	R_5	R_6	R_0	R_2	R_3	R_1
R_5	R_5	R_6	R_4	R_7	R_2	R_0	R_1	R_3
R_6	R_6	R_4	R_7	R_5	R_1	R_3	R_0	R_2
R_7	R_7	R_5	R_6	R_4	R_3	R_1	R_2	R_0

- ✅ **Closure and associativity**: The composition of two elements of S_G is also in S_G. For example, as we saw above and can verify in the table, $R_4 \circ R_1 = R_7 \in S_G$.
- ✅ **Existence of an identity element**: There is a *neutral* function, R_0, that sends a permutation to itself. This is illustrated by the highlighted row and column in the following figure.

∘	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7
R_0	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7
R_1	R_1	R_2	R_3	R_0	R_6	R_7	R_5	R_4
R_2	R_2	R_3	R_0	R_1	R_5	R_4	R_7	R_6
R_3	R_3	R_0	R_1	R_2	R_7	R_6	R_4	R_5
R_4	R_4	R_7	R_5	R_6	R_0	R_2	R_3	R_1
R_5	R_5	R_6	R_4	R_7	R_2	R_0	R_1	R_3
R_6	R_6	R_4	R_7	R_5	R_1	R_3	R_0	R_2
R_7	R_7	R_5	R_6	R_4	R_3	R_1	R_2	R_0

- ✅ **Existence of an inverse**: Every element in S_G can be mapped to the identity element, R_0, using composition with another element.

For instance all the flipping operations $R_4 \cdots R_7$ are their own inverse. The inverse of R_3 is R_1 because $R_3 \circ R_1 = R_0$, as illustrated in the following figure.

∘	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7
R_0	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7
R_1	R_1	R_2	R_3	R_0	R_6	R_7	R_5	R_4
R_2	R_2	R_3	R_0	R_1	R_5	R_4	R_7	R_6
R_3	R_3	R_0	R_1	R_2	R_7	R_6	R_4	R_5
R_4	R_4	R_7	R_5	R_6	R_0	R_2	R_3	R_1
R_5	R_5	R_6	R_4	R_7	R_2	R_0	R_1	R_3
R_6	R_6	R_4	R_7	R_5	R_1	R_3	R_0	R_2
R_7	R_7	R_5	R_6	R_4	R_3	R_1	R_2	R_0

Dihedral group

This group of symmetries is called the *dihedral* group of order 8 and is denoted D_4. For any polygon of n sides, the dihedral group with $2n$ elements is denoted D_n or D_{2n}.

Why is called diehedral?

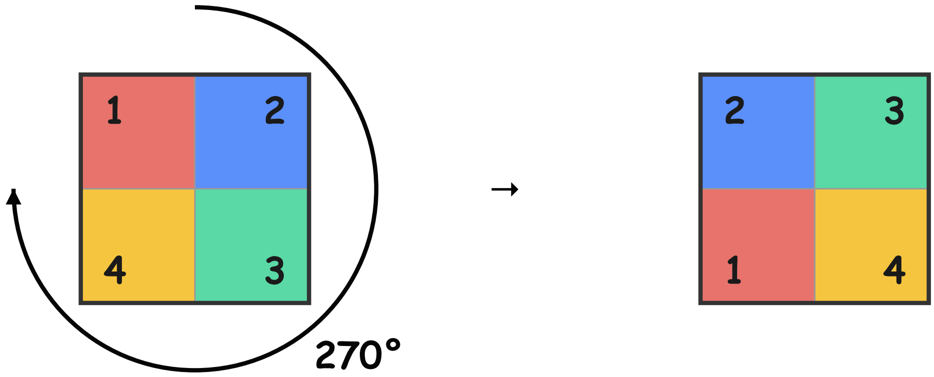
I'll leave this one as an exercise for you to do some research on.

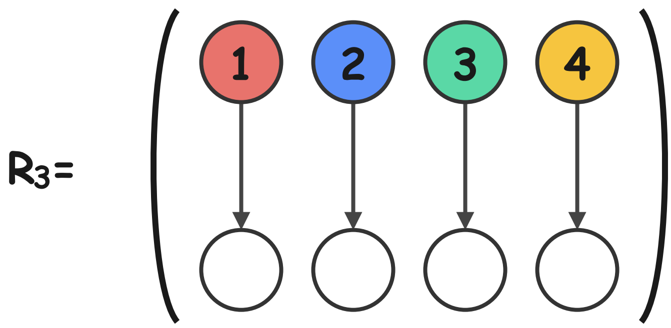
Exercises

> **Question**
>
> Why don't we list rotations that are multiples of one turn, such as 1 or 2 or 10 turns?

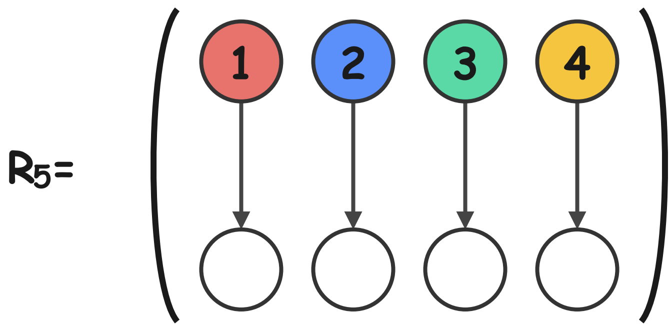
Complete the following mappings.

- R_3, rotage by 270 degrees.





- R_5: flip about axis B.



Use your paper square to compute the following operations. Use the Cayley table above to verify.

$$R_4 \circ R_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \bigcirc & \bigcirc & \bigcirc & \bigcirc \end{pmatrix} = ?$$

$$R_4 \circ R_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \bigcirc & \bigcirc & \bigcirc & \bigcirc \end{pmatrix} = ?$$

$$R_1 \circ R_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \bigcirc & \bigcirc & \bigcirc & \bigcirc \end{pmatrix} = ?$$

$$R_1 \circ R_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \bigcirc & \bigcirc & \bigcirc & \bigcirc \end{pmatrix} = ?$$

$$R_3 \circ R_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \bigcirc & \bigcirc & \bigcirc & \bigcirc \end{pmatrix} = ?$$

Credits

- Cover image by Gerardo Restrepo.
- If not explicitely cited, images (such as those in the tooltips) come from Freesvg.org and are licensed under Creative Commons CC0.
- Emoji artwork provided by Twemoji, licensed under CC-BY 4.0.

Disclaimer

After my friend Carlos Perilla helped me install Slackware on my first computer, the first software I used was LaTeX. I have loved it ever since, even if our relationship has sometimes been complicated. Over the years, I have used it for many things, from drawing circuits and graphs, to making presentations to writing love letters—no kidding. I have also written many tools that use and generate LaTeX. However, I think I spent way too much time getting things "perfect"; fine-tuning diagrams, getting the right alignment, and so on. In that regard, I'm glad to be living in the age of LLMs. I wish I had had one years ago.

All this long prolegomenon to say that I used an LLM here and there to write the code that generates some of the figures in this book. For me, it was clear from the beginning that the figures had to be generated programmatically. I reused a few libraries I had written in the past, but using the LLM allowed me to move fast and focus elsewhere. This code is available at yoyo-plots[1]. I tried to sanitize it and maintain a coherent structure, but it is still LLM code.

[1]https://github.com/twaclaw/yoyo-plots